SOME PROBLEMS ON STABILITY OF MOTION IN TOPOLOGICAL DYNAMICS

By

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By

SHAH MD. SHAMIM IMDADI

to the

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY
KANPUR

To the memory of my father

9/4/73
B

CERTIFICATE

This is to certify that the work embodied in the thesis " Some problems on stability of motion in topological dynamics " by Shah Md. Shamim Imdadi has been carried out under my supervision and has not been submitted elsewhere for a degree or diploma.

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March 1973

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SMS Imdadi

March 1973. (Shah Md. Shamim Imdadi)

CONTENTS

SYNOPSIS

'Some problems on stability of motion in topological dynamics', a thesis submitted in partial fulfilment of the requirements for the Ph.D. degree by Shah Md. Shamim Imdadi to the Department of Mathematics, Indian Institute of Technology, Kanpur.

The theory of dynamical system may be said to have begun as a special topic in the theory of ordinary differential equations with the pioneering work of Henri Poincaré in the late 19th century. Probably one of the greatest contributions Professor George Birkhoff made to mathematics was his work on the theory of dynamical systems. His work in the early part of this century formed the foundation of the modern (axiomatic) theory of topological dynamics. Afterwards quite a number of authors have studied various aspects of the theory of dynamical systems, defined on metric spaces, and in particular the theory of stability with its concrete applications to differential equations. It has long been realized that the theory of topological dynamics has many important applications to study the properties of solutions of autonomous differential equations and many interesting results have been accumulated. However until quite recently, attempts have not been made to develop and unify the theory of topological dynamics in a general topological space, and as a powerful technique in applications to nonautonomous differential equations. The objective of this investigation is (i) to obtain necessary and sufficient conditions for uniform stability, attraction and asymptotic stability of arbitrary subsets of the phase space of a dynamical system defined on uniform spaces, and (ii) to study some of the implications of the theory of topological dynamics in applications to nonautonomous differential equations.

The present thesis is divided into six chapters. The first chapter is devoted to introduction and the outline of the thesis. Chapter 2 deals with preliminaries and basic results. Definitions of uniform spaces, dynamical systems, trajectory and invariant sets, together with Kamke's lemma and other known elementary results which are used throughout the thesis, are included.

In chapter 3, the concepts of uniform stability, attraction and asymptotic stability of arbitrary subsets of the phase space are introduced in the usual way and several of their interactions have been studied for a dynamical system defined on a uniform space. These concepts are new and reduce to usual concepts for a dynamical system defined on a metric space. Further it has been shown that some of these concepts are related to Lyapunov-like behavior of appropriate real functions on the phase space. Necessary and sufficient conditions for asymptotic stability of arbitrary subsets of the phase space of a dynamical system defined on a uniform space X in terms of quasi-Lyapunov functions are obtained, without any assumptions of local compactness on X. These results are generalizations of similar results known for dynamical systems defined on a locally compact space .

Chapter 4 is devoted to a problem posed by Professor G.R.Sell. It is known that the solutions of every regular (nonautonomous) differential equation

$$x' = f(x,t), \tag{1}$$

defined on $W \times R$ with values in R^n, can be viewed as a local dynamical

system π on $W \times F^*_{co}$, where W is an open set in R^n and F^*_{co} is the closure of the space of translates of f in the compact open topology on $C(W \times R, R^n)$. The projection of π onto F^*_{co} defines a dynamical system π^* on F^*_{co}. The set of limiting equations for (1) is defined as the set of all equations

$$x' = f^*(x,t), \; f^* \in \Omega^*_f , \tag{2}$$

where Ω^*_f denote the ω-limit set of f with respect to π^*. The problem posed by G.R. Sell is the following :

"Let $f \in C(W \times R, R^n)$ be a regular function that is positively compact and assume that $f(0,t) = 0$ for $t \geq 0$. Assume also that the null solution of every limiting equation (2) is uniformly asymptotically stable. Is it true that the null solution of (1) is uniformly asymptotically stable?". A complete answer to this problem has been given in Chapter 4 which generalizes some of the results of L. Markus.

In Chapter 5, some stability properties of solutions of (1) are investigated by using the concepts of prolongation and prolongational limit sets. This has been achieved by comparing the solutions $\phi(x,f,t)$ of (1) with the corresponding motion $\pi(x,f,t)$ in $W \times F^*_{co}$. In particular, the prolongational limit set of the motion $\pi(x,f,t)$ has been compared with the prolongational limit set of the corresponding solution $\phi(x,f,t)$ to show that the latter is quasi-invariant in the sense defined by R.K.Miller. A number of results for compact solutions of the limiting equations (2) have been obtained which generalize some of the results of G.R. Sell.

Further, necessary and sufficient conditions for stability and asymptotic stability of the solutions of (1) in terms of its prolongation and prolongational limit sets have been investigated which establish a relationship between the stability of local dynamical system π and the stability of solutions of the corresponding differential equation.

In the last chapter, an attempt has been made for viewing the solutions of nonautonomous differential equations as local dynamical systems by defining the prolongational set D_f^*. Further, sufficient conditions for stability properties of solutions of a given differential equation and the corresponding set of limiting equations in terms of Lyapunov functions have been investigated. A simple example is constructed to illustrate the results.

CHAPTER 1

INTRODUCTION

1.1 HISTORICAL NOTES

In the theory of ordinary differential equations two streams of thought are mixed : analytical and geometrical (or rather topological). To this mixture one may trace many difficulties of this theory and also much of its attractiveness.

Now differential equations, more than any other part of mathematics, receive their impulsion from physics, understood in the largest sense possible. So often therefore, the solution of differential equations must come down to explicit and even numerical expressions. However all too often this can only be accomplished under very restrictive approximations. The problem therefore arises to obtain atleast some qualitative, that is, topological information about elusive solutions. Frequently also the requirements of the physicist are not for an exact, isolated solution, but for the behavior of a whole family of solutions. And this leads again to the topological behavior of the solutions. This general point of view has led in recent years to an endeavor to isolate if possible the topological from the analytical study of differential equations and has given rise to topological dynamics.

Historically, the subject of topological dynamics is an outgrowth of the qualitative theory of differential equations. To

borrow an introductory sentence from Gottschalk and Hedlund [16], "It was Poincaré who first formulated and solved problems of dynamics as problems in topology". Poincaré, followed by Bendixon, studied topological properties of solutions of autonomous ordinary differential equations in the plane. The Poincaré-Bendixon theory [15] is now a standard topic of discussion in courses on ordinary differential equations. One of the main aspects of the theory is the introduction of the concept of a trajectory. By introducing this concept of trajectory, Poincaré was able to formulate and solve problems in the theory of differential equations as topological problems.

In this fashion Poincaré paved the way for the formulation of the abstract notion of a dynamical system, which can be essentially attributed to A.A. Markov [33] and H. Whitney [49]. These two authors separately noticed that one could study the qualitative theory of families of curves (trajectories) in a suitable space X, provided that these families are somehow restricted in their possible behavior, e.g., if they are defined, as having been generated by a general one-parameter topological transformation group acting on X.

A full fledged attack on the subject was made by G.D. Birkhoff, who may truly be considered as the founder of the theory. His monograph "Dynamical Systems, Amer. Math. Soc. Colloquium Publications, Vol. 9, New York 1927" is the basis of much

of the research which came in the 1930's and 1940's.

In 1949, Nemytskii [37] wrote a survey paper on the topological problems in the theory of dynamical systems, which sums up almost all the research into the topological theory to the end of 1940's. The book "Qualitative Theory of Differential Equations" by V.V. Nemytskii and V.V. Stepanov [38] has served as a standard reference for all the major development in the theory of dynamical systems.

During the 1950's a relatively large effort went into the generalization of the concept of a dynamical system to topological transformation groups. In 1955, the book of W.H. Gottschalk and G.A. Hedlund [16] appeared, and a large body of research has appeared since in print.

Recently the basic theory has been extended by bringing in the concept of Lyapunov's stability which was absent in earlier works on dynamical systems and topological transformation groups. In this connection, T. Ura's work on the theory of prolongations and its connections with stability theory has clearly shown that a significant portion of stability theory is topological in nature and therefore belongs to the main stream of the theory of dynamical systems. An attempt in bringing in the direct method of Lyapunov was made by V.I. Zubov [54]. However, Zubov mainly carried over the previously known results and methods in differential equations to flows in metric spaces without attempting to develop an independent theory.

1.2 BRIEF REVIEW

In the theory of dynamical systems, the study of stability, prolongation, attraction, and saddle and nonsaddle sets, for a dynamical system defined on a locally compact space, has been pursued at length by Auslander, Bhatia and Seibert [2] , Auslander and Seibert [3] , Bhatia [4,5,6] , Bhatia and Szegö [11,12] , Lefschetz [32], and Ura [47] among others, and many interesting results have been accumulated. For the widest possible applicability of the theory of dynamical systems, it is of utmost importance that the phase spaces be not locally compact. In this respect, the study of dynamical systems has lagged behind. Some work on dynamical systems connected with the problems of stability theory and minimal sets and description of flows near compact invariant sets has been done by Ahmad [1] and Bhatia [7] without assuming local compactness of the phase space. In [7] , Bhatia has introduced the concepts of orbital stability, weak attraction, attraction, strong attraction, and saddle and nonsaddle sets for a dynamical system defined on a Hausdorff space. These concepts are then used to discuss several of their interactions and to discuss the flow near an arbitrary compact subset of the phase space. Of the above concepts, the concept of strong attraction is relatively new being first introduced by Bhatia and Hajek in [8] , and the notions of orbital stability, weak attraction, and attraction were also introduced in [8] , though the latter of these are well-known in ordinary differential equations and for

dynamical systems defined on locally compact spaces [5,11,12].

Bhatia and Hajek [9] and Bushaw [14] have studied various concepts associated with a dynamical system defined on a uniform space. It is shown in [9] that the notions of uniform stability of subsets and Lyapunov stability of points are related to a Lyapunov-like behavior of appropriate real functions on the phase space. This yields a characterization of stability of compact subsets of a completely regular space without any assumption of local compactness on the phase space.

Necessary and sufficient conditions for uniform stability, attraction and asymptotic stability of arbitrary subsets of the phase space of a dynamical system defined on a uniform space are investigated in Chapter 3. These concepts of uniform stability, attraction and asymptotic stability reduce to usual concepts in metric spaces. Further it is shown that the notions of asymptotic and uniform asymptotic stability are related to a Lyapunov-like behavior of appropriate real functions on the phase space. Although we confine ourselves to dynamical systems, we hope that our contribution will lead to an understanding and development of these concepts for local dynamical systems and in particular for functional and partial differential equations a large class of which define local dynamical systems on nonlocally compact Banach spaces. Also our results are marked by the absence of the assumption of local compactness on the phase space. They generalize and seem to give a

better understanding of the nature of similar results known for dynamical systems defined on locally compact spaces.

G.R. Sell [43] has shown that there is a way of viewing the solutions of a nonautonomous differential equation as a dynamical system. R.K. Miller and G.R. Sell [36] have studied Volterra integral equations within the framework of the theory of topological dynamics. These are in contrast to the theory of dynamical system which was mainly motivated from and applies to ordinary autonomous differential systems. In fact, it has been observed in the book of Nemytskii and Stepanov [38] that the solutions of a nonautonomous ordinary differential equation can be viewed as a dynamical system by imbedding the given differential equation in a higher dimensional phase space and treating the independent variable as a new coordinate. But in this construction, the new equation will not have any bounded motions, nor any periodic motions, nor any almost periodic motions, so it has the effect of destroying some of the latent structure of the original equation. Because of this fact, the theory of topological dynamics has not developed into a powerful technique in applications to nonautonomous differential equations.

However, recently G.R. Sell and R.K. Miller among others have shown that applications of topological dynamics are possible when treating nonautonomous differential equations, and these constructions do not have the defects as mentioned above. G.R.Sell [43]

has developed the basic theory for viewing the solutions of nonautonomous differential equations satisfying only the weakest hypotheses as dynamical systems. In [44], G.R. Sell has investigated the relationship between the solution of a given nonautonomous differential equation and the corresponding local dynamical system. The concept of the 'set of limiting equations', introduced by G.R. Sell, plays a central role in this investigation. Further, by using the concept of ω-limit set, he has established the basic relationship between the solutions of a given differential equation and the solutions of the corresponding limiting equations.

The aim of chapters 4 and 5 is to develop various concepts of dynamical systems and investigate some of the implications of the theory of topological dynamics in the setting of [43] and [44]. We then go on to answer an important problem on uniform asymptotic stability posed by G.R. Sell in [44]. In fact, we have proved a theorem on uniform asymptotic stability of the null solution of a system of differential equations while assuming that the null solution of a limiting equation is uniformly asymptotically stable. This generalizes a result of L. Markus [34]. By using the concepts of prolongation and prolongational limit sets for the solution of a given differential equation, we obtain some results which relate the behavior of positively compact solutions with the behavior of solutions of the corresponding limiting equations. This study includes some of the results of G.R. Sell [44]. Further, we have

given some theorems, which characterize the stability properties of solutions of a given differential equation in terms of its prolongation and prolongational limit sets. This yields in establishing some relationship between the stability of local dynamical system and the stability of solution of the corresponding differential equation.

In the last chapter, the phase space of the local dynamical system is extended by defining the prolongational set and sufficient conditions for stability of solutions of the given differential equation and the corresponding set of limiting equations in terms of Lyapunov functions have been investigated. This generalizes some of the results of G.R. Sell in [43] and [44].

1.3 OUTLINE OF THE THESIS

The purpose of the present thesis is to study the various concepts in topological dynamics with special reference to stability theory.

As concerns the formal structure of the thesis, it is divided into six chapters and each chapter is subdivided into sections. Chapters 3,4,5 and 6 form the main body of the thesis.

Chapter 1 is devoted to an introduction of the subject of topological dynamics and a brief review of results that are investigated in the present work.

Chapter 2 contains the requisite mathematical equipment such as basic definitions and elementary results in dynamical systems which are useful in our subsequent discussion.

Chapter 3 deals with the concepts of uniform stability, attraction and asymptotic stability for dynamical systems defined on uniform spaces. Various properties of weak attractor, attractor and strong attractor have been studied to relate these concepts with the concepts of uniform stability and asymptotic stability. In particular, some characterizations of uniform stability and asymptotic stability in terms of attractors are obtained which generalize some of the recent results for dynamical systems defined on locally compact spaces. In the end of the chapter, two theorems are given which characterize global asymptotic stability and global uniform asymptotic stability of arbitrary subsets of a completely regular space X without any assumption of local compactness on X, in terms of quasi-Lyapunov functions.

In Chapter 4, using the concept of the set of limiting equations for a system of differential equations, a theorem on uniform asymptotic stability of the null solution of the given system is proved. This gives a complete answer to a problem posed by G.R. Sell in [44, p.273] and generalizes a result of L. Markus [34].

Chapter 5 is devoted to the basic notions of prolongation and prolongational limit sets of solutions of a given differential

equation. Some results which relate the behavior of a positively compact solution of a given differential equation with the behavior of solutions of the corresponding limiting equations have been obtained. Further, an attempt has been made to correlate the stability properties of solutions of a given differential equation in terms of its prolongation and prolongational limit sets. This yields, in establishing a relationship between the stability of local dynamical system and the stability of solutions of the corresponding differential equation.

The phase space of local dynamical system has been extended in Chapter 6 by defining prolongational set. Further, sufficient conditions to establish a relationship between the stability property of solutions of a given differential equation and the corresponding set of limiting equations, in terms of Lyapunov functions, are investigated. A simple example is constructed to illustrate the results.

CHAPTER 2

PRELIMINARIES AND BASIC RESULTS

2.1 INTRODUCTION

The purpose of this chapter is to recall and collect the main tools that are required in our subsequent discussion. Section 2.2 contains topological preliminaries and basic results that are needed throughout the thesis. Section 2.3 introduces the fundamentals of the theory of dynamical systems. It contains essentially the definitions of dynamical systems and local dynamical systems together with the notion of trajectories, invariant sets and limit sets. The concluding section consists of some elementary concepts and basic results.

2.2 TOPOLOGICAL PRELIMINARIES

Definition 2.2.1.

A uniformity for a set X is a nonempty family $\mathcal{U}$ of subsets of $X \times X$ such that

(a) each member of $\mathcal{U}$ contains the diagonal Δ ($\Delta = \{(x,x) : x \in X\}$);

(b) if $u \in \mathcal{U}$, then $u^{-1} \in \mathcal{U}$ ($u^{-1} = \{(x,y) : (y,x) \in u\}$);

(c) if $u \in \mathcal{U}$, then $v \circ v \subset u$ for some v in $\mathcal{U}$ ($u \circ v$ stands for the set of all pairs (x,z) such that for some y it is true that $(x,y) \in v$ and $(y,z) \in u$);

(d) if u and v are members of $\mathcal{U}$, then $u \cap v \in \mathcal{U}$; and

(e) if $u \in \mathcal{U}$ and $u \subset v \subset X \times X$, then $v \in \mathcal{U}$.

Definition 2.2.2.

The topology τ of the uniformity $\mathcal{U}$, or the uniform topology, is the family of all subsets A of X such that for each x in A there is a u in $\mathcal{U}$ such that $u[x] \subset A$. (For each subset M of X the set $u[M]$ is defined to be $\{y : (x,y) \in u$ for some x in $M\}$, and if x is a point of X, then $u[x]$ is $u[\{x\}]$).

By $(X, \mathcal{U})$ we shall always mean a set X with topology τ defined by the uniformity $\mathcal{U}$ and call it as a uniform space.

A uniform space X is always completely regular, and if $\cap\{u : u \in \mathcal{U}\}$ is the diagonal Δ, then it becomes a Tychonoff space.

Definition 2.2.3.

A subfamily $\mathcal{B}$ of a uniformity $\mathcal{U}$ is a base for $\mathcal{U}$ if and only if each member of $\mathcal{U}$ contains a member of $\mathcal{B}$.

The following well-known results on uniform spaces are used in the subsequent discussion.

Theorem 2.2.1.

The interior of a subset A of X relative to the uniform topology is the set of all points x such that $u[x] \subset A$ for some u in $\mathcal{U}$.

Theorem 2.2.2.

If u is a member of the uniformity $\mathcal{U}$, then the interior of u is also a member; consequently the family of all open symmetric members of $\mathcal{U}$ is a base for $\mathcal{U}$.

Theorem 2.2.3.

The closure, relative to the uniform topology, of a subset A of X is $\bigcap\{u[A]: u \in \mathcal{U}\}$.

Theorem 2.2.4.

Each neighborhood of a compact subset A of X contains a neighborhood of the form $u[A]$ where u is a member of the uniformity $\mathcal{U}$.

Theorem 2.2.5.

If A is a subset of X, then for any $v \in \mathcal{U}$ there is a $u \in \mathcal{U}$ such that $u[\bar{A}] \subset v[A]$.

Definition 2.2.4.

Let F be a family of functions on a topological space X to a topological space Y. For each subset K of X and each subset G of Y, define $W(K,G)$ to be the set of all members of F which carry K into G, that is $W(K,G) = \{f: f[K] \subset G\}$. The family of all sets of the form $W(K,G)$, for K a compact subset of X and G open in Y, is a subbase for the compact open topology for F.

A metric for the compact open topology on the class of all continuous functions $C(W \times R, R^n)$, where W is an open subset of Euclidean n-space R^n, can be constructed as follows: Let $\{K_n\}$ be a sequence of compact sets in $W \times R$ such that $W \times R = \bigcup_{n=1}^{\infty} K_n$. For each n we construct a pseudometric

$$\rho_n = \frac{||f - g||_n}{1 + ||f - g||_n},$$

where $||f - g||_n = \sup \{|f(x,t) - g(x,t)| : (x,t) \in K_n\}$.
The required metric is given by

$$\rho(f,g) = \sum_{n=1}^{\infty} 2^{-n} \rho_n(f,g).$$

The metric ρ depends on the choice of the sequence $\{K_n\}$, however any other sequence of compact sets would generate an equivalent metric. If $K_n \subset K_{n+1}$, $n = 1,2,\ldots$, then the space $C(W \times R, R^n)$ is complete with respect to the above metric.

2.3 DYNAMICAL SYSTEMS

Let (X,d) be a metric space and let R denote the set of reals with its usual topology, R^+ and R^- the sets of nonnegative and nonpositive real numbers, respectively.

Definition 2.3.1.

A dynamical system on X is defined to be a mapping $\pi : X \times R \to X$ that satisfies the following properties :

(i) $\pi(x,0) = x$ for all $x \in X$;

(ii) $\pi(\pi(x,t),s) = \pi(x,t+s)$ for all $x \in X$ and all $t,s \in R$;

(iii) π is continuous.

The space X and the map π are respectively called the phase space and the phase map of the dynamical system.

If $M \subset X$ and $E \subset R$, then $\pi(M,E)$ stands for the set

$$\{\pi(x,t) : x \in M \text{ and } t \in E\} .$$

When M or E is a singleton we write $\pi(x,E)$ or $\pi(M,t)$ for $\pi(\{x\}, E)$ or $\pi(M, \{t\})$, respectively.

The phase map determines two other maps when one of the variables x or t is fixed. Thus for a fixed $t \in R$, the map $\pi^t : X \to X$ defined by $\pi^t(x) = \pi(x,t)$ is called a transition, and for a fixed $x \in X$, the map $\pi_x : R \to X$ defined by $\pi_x(t) = \pi(x,t)$ is called a motion through x.

We now state the following result whose proof can be found in [12] .

Theorem 2.3.1.

For each $t \in R$, the mapping π^{-t} is inverse of the mapping π^t and the mapping π^t is a topological transformation of X onto itself.

Definition 2.3.2.

For any subset M of X we define $\gamma(M)$, $\gamma^+(M)$, $\gamma^-(M)$ by

$$\gamma(M) = \pi(M,R), \quad \gamma^+(M) = \pi(M,R^+), \quad \gamma^-(M) = \pi(M,R^-).$$

If $M = \{x\}$ the corresponding sets are denoted by $\gamma(x)$, $\gamma^+(x)$, $\gamma^-(x)$ and are, respectively, called the trajectory, the positive trajectory, the negative trajectory of x (or from x) (or through x).

Definition 2.3.3.

A subset M of X is called invariant, positively invariant, or negatively invariant, if and only if $M = \gamma(M)$, $M = \gamma^+(M)$, $M = \gamma^-(M)$, respectively.

Remark 2.3.1.

For any $M \subset X$, the sets $\gamma(M)$, $\gamma^+(M)$, $\gamma^-(M)$ are always, respectively, invariant, positively invariant, negatively invariant.

Regarding invariance the following two results are well known (cf. [12]).

Lemma 2.3.1.

The boundary, closure, interior, and complement of an invariant set are invariant. The closure and interior of a positively invariant set are positively invariant. The complement of a positively invariant set is negatively invariant.

Lemma 2.3.2.

The union and intersection of a family of invariant (positively invariant) sets are invariant (positively invariant).

Definition 2.3.4.

The positive limit set (or ω-limit set) $\Omega(x)$ and the negative limit set (or α-limit set) $A(x)$ of a point x in X are defined by

$$\Omega(x) = \bigcap_{\tau} C\ell[\gamma^{+}(\pi(x,\tau))], \; A(x) = \bigcap_{\tau} C\ell[\gamma^{-}(\pi(x,\tau))],$$

where $C\ell$ denotes the closure operation on X. A point y is in $\Omega(x)$ (or $A(x)$) if and only if there is a sequence $\{t_n\}$ in R with $t_n \to +\infty$ (or $t_n \to -\infty$) and $\pi(x,t_n) \to y$.

Definition 2.3.5.

A motion $\pi(x,t)$ is said to be positively compact (or negatively compact) if $\gamma^{+}(x)$ (or $\gamma^{-}(x)$) lies in a compact subset of X. The motion is compact if $\gamma(x)$ lies in a compact subset of X.

Local Dynamical Systems

Let $I = (\alpha,\beta)$ be an open interval in R. If $\phi : I \to X$ we define the phrase " $\phi(t) \to \omega$ as $t \to$ bdy I" as follows :

(i) $I \neq R$, that is, either $\alpha \neq -\infty$ or $\beta \neq +\infty$.

(ii) If $\alpha \neq -\infty$ (or, respectively, $\beta \neq +\infty$), then for every compact subset K of X, there is a T, $\alpha < T < \beta$, such that $\phi(t) \in X - K$ for $\alpha < t \leq T$ (or, respectively, $T \leq t < \beta$).

For each point $x \in X$ let $I_x = (\alpha_x, \beta_x)$ be an open interval in R containing 0. Let

$$F = \{(x,t) \in X \times R : t \in I_x\} .$$

Definition 2.3.6.

A function $\pi : F \to X$ is said to be a local dynamical system on X if the following properties hold :

(i) $\pi(x,0) = x$, for all x in X.

(ii) If $t \in I_x$ and $s \in I_{\pi(x,t)}$, then $t + s \in I_x$ and $\pi(\pi(x,t),s) = \pi(x,t+s)$.

(iii) π is continuous.

(iv) Each interval I_x is maximal in the sense that either $I_x = R$, or $\pi(x,t) \to \omega$ as $t \to \text{bdy } I_x$.

(v) The intervals I_x are lower semicontinuous in x, that is, if $x_n \to x$, then $I_x \subset \liminf I_{x_n}$.

Remark 2.3.2.

A local dynamical system π becomes a local semi-dynamical system if we replace the domain F of π by

$$G = \{(x,t) \in X \times R^+ : 0 \leq t < \beta_x\} .$$

The sets $\gamma(x) = \{\pi(x,t) : t \in I_x\}$, $\gamma^+(x) = \{\pi(x,t) : 0 \leq t < \beta_x\}$, $\gamma^-(x) = \{\pi(x,t) : \alpha_x < t \leq 0\}$ are, respectively,

the trajectory, the positive trajectory, the negative trajectory through x. The definitions of positive invariant and invariant sets are the same as in the case of dynamical systems. Now define the sets

$$LB^+ = \{x \in X : \beta_x = +\infty\},$$
$$LB^- = \{x \in X : \alpha_x = -\infty\},$$
$$LB = LB^+ \cap LB^-.$$

We note that if LB is nonempty then $LB \times R \subset F$ and π, restricted to $LB \times R$, maps $LB \times R$ into LB. This means that $\pi : LB \times R \to LB$ is a dynamical system on LB.

If $x \in LB^+$, we define the ω-limit set $\Omega(x)$ as in Definition 2.3.4. Similarly, the α-limit set $A(x)$ is defined for all $x \in LB^-$. Also the definitions of positively compact, negatively compact and compact motions are the same as in Definition 2.3.5. Because of the maximality of I_x we see that if $\pi(x,t)$ is positively compact, then $x \in LB^+$. Similarly, if $\pi(x,t)$ is negatively compact (compact), then $x \in LB^-(x \in LB)$. The following lemmas which show the relationship between local dynamical systems on X and dynamical systems on LB are well-known [43] .

Lemma 2.3.3.

Let $\pi(x,t)$ be a positively compact motion. Then $\Omega(x)$ is nonempty, compact, and invariant. Moreover, for every $y \in \Omega(x)$, $I_y = R$.

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If $x \in LB^+$, we define the ω-limit set $\Omega(x)$ as in Definition 2.3.4. Similarly, the α-limit set $A(x)$ is defined for all $x \in LB^-$. Also the definitions of positively compact, negatively compact and compact motions are the same as in Definition 2.3.5. Because of the maximality of I_x we see that if $\pi(x,t)$ is positively compact, then $x \in LB^+$. Similarly, if $\pi(x,t)$ is negatively compact (compact), then $x \in LB^-(x \in LB)$. The following lemmas which show the relationship between local dynamical systems on X and dynamical systems on LB are well-known [43] .

Lemma 2.3.3.

Let $\pi(x,t)$ be a positively compact motion. Then $\Omega(x)$ is nonempty, compact, and invariant. Moreover, for every $y \in \Omega(x)$, $I_y = R$.

Lemma 2.3.4.

Let π be a local dynamical system on X. If there exists a positively compact motion, then LB is nonempty and the restriction of π to LB defines a dynamical system on LB.

We shall need the following formulation of continuity of π in the sequel.

Lemma 2.3.5.

Let π be a local dynamical system on X. If $\{x_n\}$ is a sequence in X and $x_n \to x$, then the sequence of functions $\{\pi(x_n,t)\}$ converge to $\pi(x,t)$, and the convergence is uniform on compact sets in I_x.

2.4 ELEMENTARY CONCEPTS AND BASIC RESULTS

We shall use nets to describe several concepts. For the basic ideas and results on nets see [28] .

Let a dynamical system π on a metric space (X,d) be given. For each $x \in X$, the transition $\pi_x\colon R \to X$ defined by $\pi_x(t) = \pi(x,t)$ is a net since R is naturally directed by the relation $\geq$ on R. The expression $\pi(x,t)$ stands for this net. For a $B \subset X$, the statements such as $\pi(x,t)$ in B ultimately or $\pi(x,t)$ in B frequently also refer to this net. Specifically, the first of these statements means that there is a $T \in R$ such that $\pi(x,t) \in B$ for all $t \geq T$; the second means that for any $T \in R$ there is a $t \geq T$ such that $\pi(x,t) \in B$, or is the same as saying that there exists $t_i \to +\infty$ such that $\pi(x,t_i) \in B$. Similar remarks are to apply to nets of the

form $\pi(B,t)$ with given $B \subset X$; these are nets mapping $t \in R$ onto the set $\pi(B,t)$. We write $\pi(x,t) \to B$ to indicate that the net $\pi(x,t)$ is ultimately in every neighborhood of B. $\pi(x,t) \xrightarrow{f} y$ denotes that the net $\pi(x,t)$ is frequently in every neighborhood of the point x.

The limit sets, prolongations, and prolongational limit sets may be defined by using the net $\pi(x,t)$ as follows :

Definition 2.4.1.

For each $x \in X$, set

$\Omega(x) = \{ y : \pi(x,t) \xrightarrow{f} y\}$,

$D^+(x) = \{y : \pi(x_i,t_i) \to y$ for some $x_i \to x$ and $t_i \in R^+ \}$, and

$J^+(x) = \{y: \pi(x_i,t_i) \to y$ for some $x_i \to x$ and $t_i \to +\infty\}$.

The sets $\Omega(x)$, $D^+(x)$ and $J^+(x)$ are called positive limit set, positive prolongation, and positive prolongational limit set of x, respectively. Similarly, negative version of these sets can be defined by using R^-.

The above sets can be defined similarly for a local semi-dynamical system. We need the following results whose proofs are given in [8] .

Lemma 2.4.1.

Let X be a locally compact space. Then for any $x \in X$, $\Omega(x)$ is nonempty and compact if and only if $C\ell\ [\gamma^+(x)]$ is compact.

Lemma 2.4.2.

Let X be a locally compact space and $x \in X$. Then $D^+(x)$ is compact if and only if $J^+(x)$ is nonempty and compact.

Theorem 2.4.1.

Let X be a Hausdorff space and $x \varepsilon X$. Let there exists a neighborhood U of x with $C\ell\, [\pi(U,R^+)]$ compact. Then $D^+(x)$ and $J^+(x)$ are both compact and connected.

Theorem 2.4.2.

Let X be a Hausdorff space. A local semi-dynamical system π is stable if and only if $C\ell[\gamma^+(x)] = D^+(x)$ for each $x \varepsilon X$.

Remark 2.4.1.

The above results are proved in [8] for local semi-dynamical systems. However, the corresponding results are also true for the case of local dynamical systems.

Lemma 2.4.3. (KAMKE [27])

Let $\{g_n\} \subset C(W \times R, R^n)$ for $n = 1,2,\ldots$, and let $g = \lim g_n$, where the convergence is in the compact open topology on C. For $n = 1,2,\ldots$, let ϕ_n be a solution of $x' = g_n(x,t)$ with $\phi_n(0) \to x_o \varepsilon W$. Then there is a subsequence of $\{\phi_n\}$ that converges to a solution ϕ of $x' = g(x,t)$ that satisfies $\phi(0) = x_o$, and the convergence is uniform on compact sets in the interval of definition of ϕ.

If, in addition, the solutions of $x' = g(x,t)$ are unique, then $\phi = \lim \phi_n$, where the convergence is uniform on compact sets in the interval of definition of ϕ.

Let $x(t) = x(t,t_o,x_o)$ be any solution of the differential system

$$x' = f(x,t),\ x(t_o) = x_o,\ t_o \geq 0\ , \tag{2.4.1}$$

where $f \in C(S_\rho \times R^+, R^n)$ and $f(0,t) = 0$ for all $t \in R^+$, S_ρ being the set $\{x \in R^n : |x| < \rho\}$.

Definition 2.4.2.

The null solution $x = 0$ of (2.4.1) is

(d_1) stable if, for each $\varepsilon > 0$, $t_0 \in R^+$, there exists a positive function $\delta = \delta(t_0, \varepsilon)$ that is continuous in t_0 for each ε such that the inequality

$|x_0| \leq \delta$ implies $|x(t)| < \varepsilon$, $t \geq t_0$;

(d_2) uniformly stable if the δ in (d_1) is independent of t_0;

(d_3) quasi asymptotically stable if, for each $\eta > 0$, $t_0 \in R^+$, there exist positive numbers $\delta_0 = \delta_0(t_0)$ and $T = T(t_0, \eta)$ such that, for $t \geq t_0 + T$ and $|x_0| < \delta_0$,

$$|x(t)| < \eta;$$

(d_4) quasi uniformly asymptotically stable if the numbers δ_0 and T in (d_3) are independent of t_0;

(d_5) asymptotically stable if (d_1) and (d_3) hold simultaneously;

(d_6) uniformly asymptotically stable if (d_2) and (d_4) hold together.

Now we shall prove the following lemma which is used throughout the discussion of Chapter 3.

Lemma 2.4.4.

Let a dynamical system π on a uniform space $(X, \mathcal{U})$ be given. Consider the motion $\pi(x,t)$ for some $x \in X$. For any $v \in \mathcal{U}$ and for any $T > 0$ we can find a $u \in \mathcal{U}$ so that, if $y \in u[x]$, then

$$\pi(y,t) \in v[\pi(x,t)]$$

for $t \in [0, T]$.

Proof. Suppose that the lemma is not true. Then there exist a $v \in \mathcal{U}$ and a $T > 0$ such that for each $u \in \mathcal{U}$ and some $y \in u[x], \tau \in [0,T]$, we have

$$\pi(y,\tau) \notin v[\pi(x,\tau)].$$

This implies that there exist a net $y_n \to x$ and a net $\{t_n\} \in [0,T]$ such that $\pi(y_n,t_n) \notin v\,[\pi(x,t_n)]$. Now, since $\{t_n\}$ is bounded, there exists a subnet $t_{n_k} \to t_o$ where $t_o \in [0,T]$. Thus, we have

$$y_{n_k} \to x,\ t_{n_k} \to t_o \quad \text{and} \quad \pi(y_{n_k},t_{n_k}) \notin v\,[\pi(x,t_{n_k})]\,,$$

which leads to a contradiction as π is continuous. This completes the proof.

CHAPTER 3

DYNAMICAL SYSTEMS IN UNIFORM SPACES

3.1 INTRODUCTION

Bhatia and Szegö [11] and Ura [47] among others have studied the concepts of stability and attractors for arbitrary subsets of the phase space of a dynamical system defined on a metric space. Lefschetz [32] and Zubov [54] obtained necessary and sufficient conditions for stability and asymptotic stability of arbitrary subsets of the phase space of a dynamical system defined on a metric space, in terms of Lyapunov-like functions. Bhatia [7] has studied the concepts of stability and attractors for a dynamical system defined on a Hausdorff space. Quite recently Bhatia and Hajek [9] obtained necessary and sufficient conditions for uniform stability of arbitrary subsets of the phase space of a dynamical system defined on a uniform space.

Our aim in this chapter is to obtain necessary and sufficient conditions for uniform stability, attraction and asymptotic stability of arbitrary subsets of the phase space of a dynamical system defined on a uniform space. The concepts of uniform stability, attraction and asymptotic stability, which are introduced here for dynamical systems defined on uniform spaces, are new and reduce to the usual concepts in metric spaces. This approach simplifies the proofs of some of the theorems which are given in [11] . Further, this study includes some

of the results of [32] and [54] as special cases. Section 3.2 is devoted to the notions of stability and uniform stability of arbitrary subsets of the phase space. In Section 3.3, concepts of weak attraction, attraction, strong attraction and uniform attraction are introduced, and several of their interactions are discussed. Section 3.4 deals with the study of asymptotic and uniform asymptotic stability. It is shown that these are related to a Lyapunov-like behavior of appropriate real functions on the phase space. This yields, in particular, Theorems 3.4.5 and 3.4.6 which characterize global asymptotic stability and global uniform asymptotic stability of arbitrary subsets of a completely regular space X without any assumption of local compactness on X.

3.2 STABILITY OF SETS

Let $(X, \mathcal{U})$ be a uniform space and π be a given dynamical system on X.

Definition 3.2.1.

A set $M \subset X$ is said to be positively stable if and only if, for each $v \in \mathcal{U}$ and $x \in M$ there exists a $u \in \mathcal{U}$ (u depends on v and x) such that

$$\pi(u[x], R^+) \subset v[M] .$$

Definition 3.2.2.

A set $M \subset X$ is said to be positively uniformly stable if and only if, for each $v \in \mathcal{U}$ there exists a $u \in \mathcal{U}$ such that

$$\pi(u[M], R^+) \subset v[M] .$$

Similarly negative stability and negative uniform stability of a set $M \subset X$ can be defined by replacing R^+ by R^-.

For simplicity, in our subsequent discussion, positively stable and positively uniformly stable are referred as stable and uniformly stable, respectively.

Now, we have the following results on stability and uniform stability of arbitrary subsets of X.

Theorem 3.2.1.

If a set $M \subset X$ is uniformly stable, then M is stable.

The proof of this theorem is obvious from the definitions of stability and uniform stability.

On the other hand, it is easy to construct examples of sets which are stable but not uniformly stable [11, p. 66].

Theorem 3.2.2.

If a set $M \subset X$ is compact, then stability of M is equivalent to uniform stability.

The proof of this theorem is a direct consequence of Theorem 2.2.4 together with definitions 3.2.1 and 3.2.2, and hence omitted.

Theorem 3.2.3.

If a closed set $M \subset X$ is stable, then it is positively invariant.

Proof. Stability of M implies that $\pi(M,R^+) \subset v[M]$ for all $v \in \mathcal{U}$. Since the closure relative to a uniform topology of a subset A of X is $\cap\{u[A] : u \in \mathcal{U}\}$ (cf. Theorem 2.2.3), we have

$$\pi(M,R^+) \subset \cap\{u[M] : u \in \mathcal{U}\} = \bar{M} = M ,$$

since M is closed. But $M \subset \pi(M,R^+)$ always holds, so that we have

$$\pi(M,R^+) = M$$

which implies that M is positively invariant. This completes the proof.

Theorem 3.2.4.

If a set $M \subset X$ is uniformly stable, then $\bar{M}$ is also uniformly stable.

Proof. Uniform stability of M implies that given any $v \in \mathcal{U}$ there exists a $u \in \mathcal{U}$ such that $\pi(u[M], R^+) \subset v[M]$.

Now for this $u \in \mathcal{U}$ there exists a symmetric $w \in \mathcal{U}$ such that $w \subset w \circ w \subset u$ (cf. Theorem 2.2.2). We claim that

$$w[\bar{M}] \subset u[M] .$$

To prove this, let $x \in w[\bar{M}]$ then there is some $\bar{m} \in \bar{M}$ such that $(\bar{m},x) \in w$. If $\bar{m} \in M$, then $(\bar{m},x) \in u$ (since $w \subset u$). This implies that $x \in u[M]$. Further, if $\bar{m}$ is a limit point of M, then $w[\bar{m}]$ will contain a point m of M. It follows that $m \in w[\bar{m}]$ and $(\bar{m},m) \in w$. Since w is symmetric, we have $(m,x) \in w \circ w \subset u$ and hence $x \in u[m]$. This shows that $x \in u[M]$. Therefore, we have

$$\pi(w[\bar{M}], R^+) \subset v[M] \subset v[\bar{M}] .$$

This proves the result.

Theorem 3.2.5.

A set $M \subset X$ is uniformly stable if and only if to each $v \in \mathcal{U}$ and $x \notin v[M]$ there exists a $u \in \mathcal{U}$ such that $\gamma^-(x)$ does not

intersect $u[M]$.

Proof. If the condition is violated, then given $v \in \mathcal{U}$ there exists a point $x \notin v[M]$ such that $\gamma^-(x)$ has some point y in $u[M]$ for each $u \in \mathcal{U}$. This implies that $\gamma^+(y)$ leaves $v[M]$, which is a contradiction to uniform stability.

Conversely, suppose that for a given $v \in \mathcal{U}$ and $x \notin v[M]$ there is a u depending on v which belongs to $\mathcal{U}$ such that $\gamma^-(x)$ is outside $u[M]$. This implies that for any point $y \in u[M]$, we have $\pi(y,R^+) \subset v[M]$. This shows that

$$\pi(u[M], R^+) \subset v[M].$$

Hence the theorem is proved.

3.3. ATTRACTORS

Let M be a subset of X.

Definition 3.3.1.

A point $x \in X$ is said to be

(i) positively weakly attracted to M if and only if the net $\pi(x,t)$, $t \in R^+$, is frequently in $v[M]$ for each $v \in \mathcal{U}$;

(ii) positively attracted to M if and only if the net $\pi(x,t)$, $t \in R^+$, is ultimately in $v[M]$ for each $v \in \mathcal{U}$;

(iii) positively strongly attracted to M if and only if for each $v \in \mathcal{U}$ there exists a $u \in \mathcal{U}$ such that the net $\pi(u[x], t)$, $t \in R^+$, is ultimately in $v[M]$.

In the subsequent discussion, the word 'positively' is omitted when referring the above definitions.

Definition 3.3.2.

Define

$A_w(M) = \{x \in X : x$ is weakly attracted to $M\}$,

$A(M) = \{x \in X : x$ is attracted to $M\}$,

and $A_s(M) = \{x \in X : x$ is strongly attracted to $M\}$.

The sets $A_w(M)$, $A(M)$ and $A_s(M)$ are called the region of weak attraction, the region of attraction and the region of strong attraction, of M, respectively.

Definition 3.3.3.

If $A_w(M)$, $A(M)$ and $A_s(M)$ are equal to $w[M]$ for some $w \in \mathcal{U}$, then M is called a weak attractor, an attractor, and a strong attractor, respectively.

Definition 3.3.4.

A set $M \subset X$ is a uniform attractor if and only if there exists a $u \in \mathcal{U}$ such that, for each $x \in u[M]$, $\pi(x,t)$ is ultimately in every $v[M]$, $v \in \mathcal{U}$, uniformly in x.
The set $u[M]$ is known as the region of uniform attraction of M and is denoted by $A_u(M)$.

Remark 3.3.1.

It is obvious from the above definitions that

$$A_u(M) \subset A_s(M) \subset A(M) \subset A_w(M).$$

Theorem 3.3.1.

For any set $M \subset X$, the sets $A_w(M)$, $A(M)$, $A_s(M)$ and $A_u(M)$ are invariant.

Proof. First we prove that the set $A(M)$ is invariant. Let $x \in A(M)$. We have to show that for any $T \in R$,

$$\pi(x,T) \in A(M).$$

Since $x \in A(M)$, we have, from the definition, for any $u \in \mathcal{U}$,

$$\pi(x,t) \in u[M] \text{ as } t \to +\infty.$$

Hence from group axiom, we get

$$\pi(x,t+T) \in u[M] \text{ as } t+T \to +\infty.$$

This implies that $\pi(x,T) \in A(M)$ and hence the set $A(M)$ is invariant. The proof for $A_w(M)$ is similar and hence omitted.

To prove that $A_s(M)$ is invariant, let $x \in A_s(M)$ and $T \in R$. Let v be any member of $\mathcal{U}$. Since x is strongly attracted to M, by definition, there exists a $u \in \mathcal{U}$ such that $\pi(u[x],t)$ is ultimately in $v[M]$. By Theorem 2.3.1, $\pi^T(u[x])$ is a neighborhood of $\pi(x,T)$. Obviously, this neighborhood $\pi^T(u[x])$ of $\pi(x,T)$ has the property that $\pi(\pi^T(u[x]), t)$ is ultimately in $v[M]$, which implies that $\pi(x,T)$ is strongly attracted to M. Hence the set $A_s(M)$ is invariant. The proof for $A_u(M)$ is similar and hence omitted.

Theorem 3.3.2.

If a set $M \subset X$ is a weak attractor or an attractor or a strong attractor or a uniform attractor, then the corresponding

region of attraction is an open invariant set.

Proof. From Theorem 3.3.1, it follows that the sets $A_w(M), A(M)$, $A_s(M)$ and $A_u(M)$ are invariant. Therefore, it is enough to show that they are open.

We shall first prove that $A(M)$ is open. Since M is an attractor, by definition, there is a $w \in \mathcal{U}$ such that $A(M) = w[M]$. Now, for this w, there is a $v \in \mathcal{U}$ such that $v \subset w$ and v is an open symmetric neighborhood of the diagonal Δ. This implies that $v[M]$ is open (cf. [28]). Thus for any $x \in A(M)$ and $t \geq 0$, if $\pi(x,t) \in v[M]$, then $\pi^{-t}(v[M])$ is an open neighborhood of x, due to Theorem 2.3.1. Further, since $\pi(y,t) \in v[M]$ for each $y \in \pi^{-t}(v[M])$, we conclude that $\pi^{-t}(v[M]) \subset A(M)$, due to the invariant property of $A(M)$. Hence $A(M)$ is open. The proof for $A_w(M)$ is similar and hence omitted.

To prove that $A_s(M)$ is open, since M is a strong attractor, it follows from the definition that there is a $w \in \mathcal{U}$ such that $w[M] = A_s(M)$. For this $w \in \mathcal{U}$ there exists a $v \in \mathcal{U}$ such that v is symmetric open neighborhood of the diagonal Δ and also $v \subset w$ and hence $v[M]$ is open (cf. [28]). Now, suppose that $x \in A_s(M)$, which implies that there exists a $u \in \mathcal{U}$ such that $\pi(u[x], t)$ is ultimately in $v[M]$. This shows that there exists a $T \in R$ such that $\pi(x,T) \in v[M]$. By Theorem 2.3.1, we know that $\pi^{-T}(v[M])$ is an open neighborhood of x and hence $\pi^{-T}(v[M]) \subset w[M] = A_s(M)$, due to the invariant property of $A_s(M)$. This implies that the set $A_s(M)$ is open. Similarly, we can prove that $A_u(M)$ is an open set.

Corollary 3.3.1.

If a set $M \subset X$ is

(i) a uniform attractor, then $A_u(M) = A_w(M)$;

(ii) a strong attractor, then $A_s(M) = A_w(M)$;

(iii) an attractor, then $A(M) = A_w(M)$.

The proof is direct and simple, hence omitted.

Theorem 3.3.3.

If a set $M \subset X$ is positively invariant and $M \subset A_s(M)$, then M is stable.

Proof. Let $u[M]$ be a neighborhood of M for some $u \in \mathcal{U}$. Since $M \subset A_s(M)$, for every $x \in M$ there exists a $v \in \mathcal{U}$ and a $T \in R^+$ such that

$$\pi(v[x], t) \subset u[M] \quad \text{for} \quad t > T.$$

From the positively invariant property of M and the continuity axiom, there exists a $w \in \mathcal{U}$ such that

$$\pi(w[x], t) \subset u[M] \quad \text{for} \quad 0 \leq t \leq T.$$

Thus, by setting $v' = v \cap w$, we have $v' \in \mathcal{U}$ and

$$\pi(v'[x], t) \subset u[M]$$

for $t \in R^+$. Hence, for a given $u \in \mathcal{U}$ and $x \in M$ there exists a $v' \in \mathcal{U}$ (v' depending on u and x) such that

$$\pi(v'[x], R^+) \subset u[M].$$

Thus M is stable. This completes the proof.

Remark 3.3.2.

A similar theorem is proved in [11] by assuming that the set M is positively invariant closed set which is uniformly attracting. Obviously the assumptions of Theorem 3.3.3 are less restrictive.

Theorem 3.3.4.

Let M be a subset of X. If M is uniformly stable and a weak attractor, then the set M is a strong attractor.

Proof. Since M is a weak attractor it is sufficient to show that $A_w(M) \subset A_s(M)$. Let $x \varepsilon A_w(M)$ and $v\,[M]$ be any open neighborhood of M, where $v \varepsilon \mathcal{U}$. Since M is uniformly stable, there exists an open $u \varepsilon \mathcal{U}$ such that

$$\pi(u\,[M]\,,\, R^+) \subset v\,[M]\,. \tag{1}$$

Further, $x \varepsilon A_w(M)$ implies that there is a $\tau \varepsilon R^+$ such that

$$\pi(x, \tau) \varepsilon\, u\,[M]\,.$$

Let $\pi(x, \tau) = y$. Since $u\,[M]$ is open and $y \varepsilon u\,[M]$, it follows that there is a $w \varepsilon \mathcal{U}$ such that

$$w\,[y] \subset u\,[M]\,.$$

Therefore, from (1), we have

$$\pi(w\,[y]\,,\, R^+) \subset v\,[M]\,.$$

On the other hand, we know that $\pi^{-\tau}(w\,[y])$ is a neighborhood of x and

$$\pi^{-\tau}(w\,[y]\,) \subset A_w(M)\,.$$

This shows that the net $\pi(\pi^{-\tau}(w\,[y]\,), t)$ is ultimately in $v\,[M]$,

which implies that $x \in A_s(M)$. Hence the theorem is proved.

Theorem 3.3.5.

A compact subset M of X is uniformly stable if and only if $\bar{M}$ is positively invariant and $M \subset A_s(M)$.

Proof. The necessity follows directly from the definition of uniform stability and Theorems 3.2.3 and 3.2.4.

To prove sufficiency, let v be an arbitrary member of $\mathcal{U}$. By Theorem 2.2.5, there is a $v^* \in \mathcal{U}$ such that

$$v^* [\bar{M}] \subset v [M] . \tag{1}$$

Now, since $M \subset A_s(M)$, for a given $x \in M$ and $v^* \in \mathcal{U}$ there exist $w \in \mathcal{U}$ and $T \geq 0$ such that

$$\pi(w [x] ,t) \subset v^* [M] \text{ for all } t \geq T. \tag{2}$$

Further, by Lemma 2.4.4, given $v^* \in \mathcal{U}$ and $T > 0$, there exists a $w^* \in \mathcal{U}$ such that $y \in w^* [x]$ implies that

$$\pi(y,t) \in v^* [\pi(x,t)]$$

for $t \in [0,T]$. Since $\bar{M}$ is positively invariant, we have

$$\pi(w^* [x] ,t) \subset v^* [\bar{M}] \text{ for } t \in [0,T] . \tag{3}$$

Consider $W = w \cap w^*$. Obviously $W \in \mathcal{U}$ and from (2) and (3), we have

$$\pi(W [x] , R^+) \subset v^* [\bar{M}] .$$

Now, let

$$N = \cup \{ W [x] : x \in M \}.$$

Then, we have

$$\pi(N,R^+) \subset v^*[\bar{M}].$$

Since M is compact and N is a neighborhood of M, there exists a $u \in \mathcal{U}$ such that $u[M] \subset N$ (cf. Theorem 2.2.4). Hence, using (1), we have

$$\pi(u[M], R^+) \subset v[M] .$$

This proves that M is uniformly stable.

Theorem 3.3.6.

A compact subset M of X is uniformly stable and a weak attractor if and only if $\bar{M}$ is positively invariant and M is a strong attractor.

Proof. Let M be uniformly stable and a weak attractor. Then $\bar{M}$ is positively invariant because of Theorems 3.2.3 and 3.2.4, and M is a strong attractor follows from Theorem 3.3.4.

Conversely, let $\bar{M}$ be positively invariant and M be a strong attractor. Then, by Corollary 3.3.1, M is a weak attractor. Further, the uniform stability of M follows from Theorem 3.3.5 as $\bar{M}$ is positively invariant and $M \subset A_s(M)$. This completes the proof.

Theorem 3.3.7.

Let M be a subset of X. If $\bar{M}$ is positively invariant and M is a uniform attractor, then M is uniformly stable.

Proof is similar to the proof of Theorem 3.3.6 and hence omitted.

3.4 ASYMPTOTIC AND UNIFORM ASYMPTOTIC STABILITY

Definition 3.4.1.

A set $M \subset X$ is said to be

(i) asymptotically stable if and only if it is uniformly stable and an attractor,

(ii) globally asymptotically stable if and only if it is asymptotically stable and $A(M) = X$.

Definition 3.4.2.

A set $M \subset X$ is said to be

(i) uniformly asymptotically stable if and only if it is uniformly stable and a uniform attractor.

(ii) globally uniformly asymptotically stable if and only if it is uniformly asymptotically stable and $A_u(M) = X$.

Now, since a strong attractor is an attractor, Theorem 3.3.6 gives the following characterization of asymptotic stability of compact sets.

Theorem 3.4.1.

A compact subset M of X is asymptotically stable if and only if $\bar{M}$ is positively invariant and M is a strong attractor.

Theorem 3.4.2.

A set $M \subset X$ is uniformly asymptotically stable if and only if $\bar{M}$ is positively invariant and M is a uniform attractor.

Proof. Let M be a uniformly asymptotically stable subset of X. Then, by Definition 3.4.2, M is uniformly stable and a uniform attractor. Since M is uniformly stable, from Theorems 3.2.3 and 3.2.4, $\bar{M}$ is positively invariant. The converse of the theorem follows from Theorem 3.3.7. This completes the proof.

Theorem 3.4.3.

If a compact subset M of a Tychonoff space X is uniformly asymptotically stable then there exists a $u \in \mathcal{U}$ such that $u[M]$ is free from complete trajectories other than those in M itself.

Proof. Suppose that the condition does not hold, so that every $v[M]$ for $v \in \mathcal{U}$ contains a complete trajectory. Since M is uniformly asymptotically stable so it is uniformly stable and there is a $w \in \mathcal{U}$ such that, for each $x \in w[M]$, $\pi(x,t)$ is ultimately in every $v[M]$, $v \in \mathcal{U}$, uniformly in x. Uniform stability of M implies that there is a $u \in \mathcal{U}$ such that

$$\pi(u[M], R^+) \subset w[M] ,$$

and from uniform asymptotic stability of M it follows that there is a $T \in R^+$ such that if $x \in w[M]$ then

$$\pi(x, t) \in u[M]$$

for all $t \geq T$. Now suppose that the complete trajectory which is contained in $w[M]$ is γ. Since M is a compact subset of a Tychnoff space, we can choose u such that $u[M]$ does not contain a certain point y of γ. Hence, we have $z = \pi(y,-T) \in w[M]$, because γ is contained in $w[M]$ and $z \in \gamma$. Thus $\pi(z,T) = y \in u[M]$, which is a contradiction. This proves the theorem.

Theorem 3.4.4.

A uniformly stable strong attractor M with compact region of strong attraction is uniformly asymptotically stable.

The proof is direct and simple and hence omitted.

Now, we shall obtain necessary and sufficient conditions for global asymptotic stability and global uniform asymptotic stability of arbitrary subsets of X, in terms of quasi-Lyapunov functions (cf. [9]).

Definition 3.4.3.

A mapping $\phi : X \to R^+$ is called a quasi-Lyapunov function (to π on X) if and only if ϕ is continuous, and to each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\phi(x) < \delta \text{ implies } \phi(\pi(x,t)) < \varepsilon$$

for all $t \in R^+$.

Theorem 3.4.5.

A set $M \subset X$ is globally asymptotically stable if and only if for each $v \in \mathcal{U}$ there exists a uniformly continuous quasi-Lyapunov function $\phi_v : X \to [0,1]$ such that

$$\phi_v | M = 0, \ \phi_v | (X - v[M]) = 1,$$

and for $x \in X, \phi_v(\pi(x,t)) \to 0$ as $t \to +\infty$.

Proof. Suppose that M is globally asymptotically stable. This implies that M is uniformly stable and $A(M) = X$. Since M is uniformly stable, for each $v \in \mathcal{U}$ there is a uniformly continuous quasi-Lyapunov function $\phi_v : X \to [0,1]$ such that

$$\phi_v | M = 0 \quad \text{and} \quad \phi_v | (X - v[M]) = 1 \quad [9, \text{p. } 16] .$$

Now, take any $\varepsilon > 0$ and consider the set

$$u = \{(x,y) : |\phi_v(x) - \phi_v(y)| < \varepsilon\}.$$

Since ϕ_v is uniformly continuous, obviously $u \varepsilon \mathcal{U}$. This implies that, if $y \varepsilon u[M]$ then

$$|\phi_v(y)| < \varepsilon \quad , \text{as} \quad \phi_v | M = 0 \quad .$$

Now, let $x \varepsilon X$. As $A(M) = X$, there is a $T > 0$ such that $\pi(x,t) \varepsilon u[M]$ for $t \geq T$. This in turn implies that

$$|\phi_v(\pi(x,t))| < \varepsilon$$

for $t \geq T$. Hence $\phi_v(\pi(x,t)) \to 0$ as $t \to +\infty$.

Conversely, suppose that for each $v \varepsilon \mathcal{U}$ there exists a uniformly continuous quasi-Lyapunov function $\phi_v : X \to [0,1]$ satisfying the above conditions. Obviously M is uniformly stable [9, p. 16]. So we have only to show that $A(M) = X$. Consider any $x \varepsilon X$ and any $u \varepsilon \mathcal{U}$. For this u, there is a uniformly continuous quasi-Lyapunov function $\phi_u : X \to [0,1]$ such that

$$\phi_u | M = 0, \quad \phi_u | (X - u[M]) = 1$$

and $\phi_u(\pi(x,t)) \to 0$ as $t \to +\infty$. Now suppose that $\pi(x,t)$ is not ultimately in $u[M]$. Then for any $T \varepsilon R^+$ there is a $\tau > T$ such that

$$\pi(x, \tau)) \notin u[M] \quad .$$

This implies that

$$\phi_u(\pi(x,\tau)) = 1 .$$

Thus, it follows that $\phi_u(\pi(x,t)) \not\to 0$ as $t \to +\infty$, which is a contradiction. Hence $\pi(x,t)$ is ultimately in $u[M]$. Since x and u are arbitrary, we have $A(M) = X$. This completes the proof.

Theorem 3.4.6.

A set $M \subset X$ is globally uniformly asymptotically stable if and only if for each $v \in \mathcal{U}$ there exists a uniformly continuous quasi-Lyapunov function $\phi_v : X \to [0,1]$ such that

$$\phi_v|M = 0, \quad \phi_v|(X-v[M]) = 1,$$

and $\phi_v(\pi(x,t)) \to 0$ as $t \to +\infty$, uniformly in x.

The proof is similar to the proof of Theorem 3.4.5 and hence omitted.

CHAPTER 4

LIMITING EQUATIONS

4.1 INTRODUCTION.

The concept of the 'set of limiting equations' of a given differential equation has been introduced by G.R. Sell [44]. The notion of 'asymptotically autonomous differential equations' introduced by L. Markus [34] can be described as those differential equations for which the set of limiting equations consists of a single point. In [44], G.R. Sell has proved a theorem on asymptotic stability of the null solution of a given differential equation while assuming that the null solution of the given differential equation is uniformly stable and null solution of every limiting equation is asymptotically stable (in a uniform sense). However, as pointed out in a remark [44, p. 273] and [45, p. 536], his theorem is not generalizing a result of L. Markus [34, Theorem 2]. The aim of this chapter is to prove a theorem on uniform asymptotic stability which generalizes the result of L. Markus [34, Theorem 2].

4.2 CONSTRUCTION OF DYNAMICAL SYSTEM

Throughout this chapter, we follow the same notation as in [43] and [44].

Let W be an open set in R^n, Euclidean n-space. The Euclidean norm on R^n will be denoted by $|x|$. Let $C = C(W \times R, R^n)$ denote

the set of all continuous functions f defined on $W \times R$ with values in R^n.

Definition 4.2.1.

A function $f: W \times R \to R^n$ is said to be admissible if and only if

(i) f is continuous, and

(ii) the solutions of the differential equation $x' = f(x,t)$ are unique.

By the second condition we mean that given any point (x_o, t_o) in $W \times R$, there is precisely one solution ϕ of $x' = f(x,t)$ that satisfies $\phi(t_o) = x_o$.

Definition 4.2.2.

We shall say that an admissible function f in $C(W \times R, R^n)$ satisfies the global existence property if and only if every solution of $x' = f(x,t)$ can be continued for all t in R.

It is evident that if f is an admissible function, then every translate f_τ of f (where $f_\tau(x,t) = f(x,t+\tau)$) is an admissible function. Also, if f satisfies the global existence property, then so does each f_τ.

Let $F = \{f_\tau : \tau \in R\}$ be the space of translates of f, then F is a subset of C. Now let f be an admissible function and consider the space of translates F in the compact open topology on C. Let $F^*_{co} = C\ell\, F$ (that is, the closure in the compact open topology) be the hull of f.

Definition 4.2.3.

A function f is said to be regular if and only if every function f^* in the hull F^*_{co} is admissible.

The following result is due to G.R. Sell [43].

Theorem 4.2.1.

The mapping $\pi^* : C \times R \to R$, defined by $\pi^*(f, \tau) = f_\tau$, defines a dynamical system on C, when C has the compact open topology. Each set $F = \{f_\tau : \tau \in R\}$ is a trajectory of π^*.

It is clear that the restriction of π^* to $F^*_{co} \subset C$ is also a dynamical system.

Definition 4.2.4.

The motion f_t is said to be positively compact if and only if the closure of $\{\pi^*(f,t) : t \geq 0\}$ lies in a compact subset of F^*_{co}.

Definition 4.2.5.

Let $f \in C$ and let F^*_{co} be the hull of f (Neither regularity nor admissibility of f will be important here). Let $\pi^*(f,t) = f_t$ be the flow on F^*_{co} and let $\Omega^*(f)$ denote the ω-limit set of f in this flow. If the ω-limit set $\Omega^*(f)$ of f in F^*_{co} is nonempty, then we say that the set of limiting equations for

$$x' = f(x,t) \tag{4.2.1}$$

is the set of all differential equations of the form

$$x' = f^*(x,t), \tag{4.2.2}$$

where $f^* \varepsilon \Omega^*(f)$.

4.3 UNIFORM ASYMPTOTIC STABILITY

In Theorems 4 and 5 of [44], G.R. Sell has proved that if the equation (4.2.1), with the assumption that f is a regular function and $f(0,t) = 0$ for all $t \geq 0$, has a 'stable' solution, then the limiting equations (4.2.2) has the same property. The problem of reversing these roles is a bit delicate. That is, if we assume some stability properties of the solutions of (4.2.2), then it is generally harder to derive results about the given equation (4.2.1). Consider the following example.

Example 4.3.1.

The solutions of the linear equation

$$x' = \frac{x}{t+1} \qquad (t \geq 0)$$

are not stable. However, the solutions of the limiting equation (there is only one) $x' = 0$ are uniformly stable. (Although the above equation is defined only for $t \geq 0$, one can easily extend it for $t < 0$. This would not change the limiting equation.)

However, G.R. Sell has proved the following result.

Theorem 4.3.1 [44, *Theorem 6*].

Let $f \varepsilon C(W \times R, R^n)$ be a regular function with $f(0,t) = 0$ $(t \geq 0)$ and assume that the motion f_t is positively compact, in the compact open topology. If

(i) the null solution of (4.2.1) is uniformly stable, and

(ii) the null solution of every limiting equation (4.2.2) is asymptotically stable (in a uniform sense), that is $|\phi(x,f^*,t)| \to 0$ as $t \to +\infty$ whenever $|x| \leq a$ and $f^* \in \Omega^*(f)$,

then the null solution of (4.2.1) is asymptotically stable.

Now, we shall prove the following main result.

Theorem 4.3.2.

Let $f \in C(W \times R, R^n)$ be a regular function with $f(0,t) = 0$ $(t \geq 0)$. If there exists a function f^* in $\Omega^*(f)$ such that the null solution of (4.2.2) is uniformly asymptotically stable, then the null solution of the given equation (4.2.1) is uniformly asymptotically stable.

Proof. Let $\phi(x_0,f^*,t)$ be any solution of (4.2.2) with $\phi(x_0,f^*,0) = x_0$. Let $\varepsilon > 0$ be given. Since the null solution of (4.2.2) is uniformly asymptotically stable, given $\varepsilon/2$ there exists a $\delta_1 = \delta_1(\varepsilon/2) > 0$ (without any loss of generality we can suppose that $\delta_1 < \varepsilon/2$) such that the inequality $|x_0| < \delta_1$ implies

$$|\phi(x_0,f^*,t)| < \frac{\varepsilon}{2} \tag{4.3.1}$$

for $t \geq 0$, and there exists a $\delta_0 > 0$, such that for every $\eta > 0$ there exists a $T = T(\eta) > 0$ such that the inequality $|x_0| < \delta_0$ implies that

$$|\phi(x_0, f^*, t)| < \eta \qquad (4.3.2)$$

for $t \geq T$. Choose $\delta = \min\ (\delta_1, \delta_0)$. From (4.3.1) and (4.3.2), we have

$$|\phi(x_0, f^*, t)| < \frac{\varepsilon}{2} \qquad (4.3.3)$$

for $t \geq 0$, whenever $|x_0| < \delta$, and for $\delta/2$ there exists a $T = T(\delta/2)$ such that the inequality $|x_0| < \delta$ implies

$$|\phi(x_0, f^*, t)| < \frac{\delta}{2} \qquad (4.3.4)$$

for $t \geq T$. Now, by Lemma 2.4.3, given $f^* \varepsilon\ \Omega^*_f$ and $\delta/2$ there is a $d = d(\delta/2) > 0$ such that

$$|\phi(x_0, f^*, t) - \phi(x_0, g, t)| < \frac{\delta}{2}\text{, for}$$

$t \varepsilon [0, T]$, $|x_0| < \delta$, whenever $\rho(f^*, g) < d$, where ρ is any metric which generates the compact open topology on F^*_{co}. By the definition of $\Omega^*(f)$ we can find a translate f_τ, $\tau > 0$, such that $\rho(f^*, f_\tau) < d$. Therefore

$$|\phi(x_0, f^*, t) - \phi(x_0, f_\tau, t)| < \frac{\delta}{2} \qquad (4.3.5)$$

for $t \varepsilon [0, T]$, $|x_0| < \delta$. Combining (4.3.3) and (4.3.5), we get

$$|\phi(x_0, f_\tau, t)| \leq |\phi(x_0, f^*, t)| + |\phi(x_0, f_\tau, t) - \phi(x_0, f^*, t)|$$

$$< \frac{\varepsilon}{2} + \frac{\delta}{2}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

That is,

$$|\phi(x_0, f_\tau, t)| < \varepsilon \qquad (4.3.6)$$

for $t \in [0,T]$, $|x_0| < \delta$.

Now let $x_1 = \phi(x_0, f_\tau, T)$, then from (4.3.4) and (4.3.5), we have

$$|x_1| \leq |\phi(x_0,f^*,T)| + |x_1-\phi(x_0,f^*,T)|$$

$$< \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Therefore, from (4.3.6) it follows that

$$|\phi(x_1,f_\tau,t)| < \varepsilon$$

for $t \in [0,T]$, which in turn implies that

$$|\phi(x_0,f_\tau,t)| < \varepsilon$$

for $t \in [0,2T]$, $|x_0| < \delta$. Similarly, if we suppose $x_2 = \phi(x_1,f_\tau,T)$, then

$$|x_2| \leq |\phi(x_1,f^*,T)| + |x_2 - \phi(x_1,f^*,T)|$$

$$< \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

and again, from (4.3.6), it follows that

$$|\phi(x_0,f_\tau,t)| < \varepsilon$$

for $t \in [0,3T]$, $|x_0| < \delta$. Now, let m be a positive integer and assume that

$$|\phi(x_0,f_\tau,t)| < \varepsilon$$

for $t \in [0,mT]$, $|x_0| < \delta$, and assume that

$$|\phi(x_0,f_\tau,mT)| < \delta.$$

Let $x_m = \phi(x_o, f_\tau, mT)$, then from (4.3.3) and (4.3.5), we have

$$|\phi(x_m,f_\tau,t)| \leq |\phi(x_m,f^*,t)| + |\phi(x_m,f_\tau,t) - \phi(x_m,f^*,t)|$$

$$< \frac{\varepsilon}{2} + \frac{\delta}{2}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon ,$$

for $t \varepsilon [0,T]$. Therefore, $\phi(x_o,f_\tau,t)$ can be continued to the interval $[mT,(m + 1) T]$ on which

$$|\phi(x_o,f_\tau,t)| < \varepsilon.$$

Let $x_{m+1} = \phi(x_o,f_\tau,(m + 1)T)$, then from (4.3.4) and (4.3.5) and the fact that $|x_m| < \delta$, we have

$$|x_{m+1}| \leq |\phi(x_m,f^*,T)| + |x_{m+1} - \phi(x_m,f^*,T)|$$
$$< \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Thus, by induction, $|\phi(x_o,f_\tau,t)| < \varepsilon$, whenever $|x_o| < \delta$, on every interval $[mT,(m + 1) T]$ and hence on $[0, +\infty)$. Since δ is independent of τ, we have, for a given $\varepsilon > 0$ there exist a $\delta = \delta(\varepsilon) > 0$ and a $\tau = \tau(\varepsilon) \geq 0$ such that

$$|\phi(x_o,f,t)| < \varepsilon, \text{ for}$$

$t \geq \tau$, provided $|x_o| < \delta$. Now from the continuity of solutions with respect to the initial values and the uniqueness of solutions, it follows that the null solution of (4.2.1) is uniformly stable. For the rest of the proof, choose $\hat{\delta}_o = \delta_o$ and fix $|x_o| < \hat{\delta}_o$. Let $\eta > 0$ be given. Choose $\bar{\delta}(\eta) = \min(\delta_1(\eta/2), \hat{\delta}_o)$, $0 < \bar{\delta} < \eta$, and $T_1(\eta) = T(\bar{\delta}/2)$. Let $y_o = \phi(x_o,f_\tau,T_1)$, $\tau = \tau(\eta) \geq 0$. Then

$$|y_0| \le |\phi(x_0,f^*,T_1)| + |\phi(x_0,f_\tau,T_1) - \phi(x_0,f^*,T_1)|$$
$$< \frac{\bar\delta}{2} + \frac{\bar\delta}{2} = \bar\delta .$$

Thus by the first part of the proof,

$$|\phi(y_0,f,t)| < \eta$$

for $t \ge \tau$, whenever $|y_0| < \bar\delta(\eta)$. Now by the uniqueness of solutions, it follows that

$$|\phi(x_0,f,t)| < \eta$$

for $t \ge T^*$, whenever $|x_0| < \hat\delta_0$, where $T^* = T^*(\eta) = \tau + T_1$. This completes the proof of the theorem.

Remark 4.3.1.

Observe that in Theorem 4.3.1, G.R. Sell assumed that the motion f_t is positively compact and the null solution of every limiting equation (4.2.2) is asymptotically stable (in a uniform sense) to prove that the null solution of (4.2.1) is asymptotically stable. Clearly, Theorem 4.3.1 is not generalizing a result of L. Markus [34, Theorem 2] as remarked in [44, p. 273] and [45, p. 536]. It is apt to remark here that in Theorem 4.3.2, we merely assume that the null solution of (4.2.2) for some f^* in $\Omega^*(f)$ is uniformly asymptotically stable to prove that the null solution of (4.2.1) is uniformly asymptotically stable. Obviously, this would generalize the result of L. Markus [34, Theorem 2] for asymptotically autonomous equations.

CHAPTER 5

PROLONGATION AND PROLONGATIONAL LIMIT SETS

5.1 INTRODUCTION

Quite recently G.R. Sell [43,44] has shown that there is a way of viewing the solutions of nonautonomous differential equations as dynamical systems. This view point is very general and includes all differential equations satisfying only the weakest hypotheses. In [44], by introducing the concept of the 'set of limiting equations' for a given differential equation, he has investigated some of the implications of the theory of topological dynamics in this setting. Further, by using the concept of ω-limit set in [44], G.R. Sell has established the basic relationship between the solutions of a given differential equation and the solutions of the corresponding limiting equations. His results generalize some of the recent works of L. Markus [34] and R.K. Miller [35].

Our aim in this chapter is to introduce the concepts of prolongation and prolongational limit sets for the solution of a given differential equation in the usual way and study the properties of these sets. Section 5.2 deals with definitions and basic lemmas. In Section 5.3, we obtain some results which relate the behavior of a positively compact solution of the given differential equation with the behavior of solutions of the corresponding limiting equations.

This study includes some of the results of G.R. Sell [44] . It is well known that the concepts of prolongation and prolongational limit sets play an important role in the stability theory. In Section 5.4, we shall prove some theorems which characterize the stability properties of solutions of a given differential equation in terms of its prolongation and prolongational limit sets. Further we establish a relationship between the stability of local dynamical system π and the stability of solutions of the corresponding differential equation.

5.2 DEFINITIONS

Throughout this chapter we shall follow the definitions and notations of Section 4.2.

Let $f \in C(W \times R, R^n)$ be an admissible function and let $\phi(t) = \phi(x,f,t)$ be the solution of (4.2.1) that satisfies $\phi(x,f,0) = x$. Let $I_{(x,f)} = (\alpha, \beta)$ be the maximal interval of definition of ϕ .

Definition 5.2.1.

The solution ϕ is said to be positively compact if and only if the set $\{\phi(t) : 0 \leq t < \beta\}$ lies in a compact set in W.

Negative compactness and compactness are defined similarly. Since $I_{(x,f)}$ is maximal, it follows that $\beta = +\infty$, whenever ϕ is positively compact. If ϕ is compact, then $I_{(x,f)} = R$, that is, ϕ is defined for all t in R.

Let $f \in C(W \times R, R^n)$ be a regular function. Then the mapping [43, Theorem 8]

$$\pi(x,f;t) = (\phi(x,f,t),f_t)$$

defines a local dynamical system on $W \times F^*_{co}$, where F^*_{co} is the hull of f in the compact open topology on $C(W \times R, R^n)$. It is clear that the motion $\pi(x,f;t)$ is defined for all $t \geq 0$ if and only if the solution $\phi(x,f,t)$ is defined for all $t \geq 0$.

For $(x,f) \in W \times F^*_{co}$, $\gamma(x,f) = \{\pi(x,f;t) : t \in R\}$ is the trajectory through (x,f). Similarly $\gamma^+(x,f) = \{\pi(x,f;t): t \geq 0\}$ and $\gamma^-(x,f) = \{\pi(x,f;t) : t \leq 0\}$ are, respectively, the positive and negative semitrajectories through (x,f).

The limit sets, prolongations, and prolongational limit sets may now be defined as follows.

Definition 5.2.2.

For each $(x,f) \in W \times F^*_{co}$, let

$$\Omega(x,f) = \{(y,g) : \pi(x,f;t_i) \to (y,g) \text{ for some sequence } t_i \to +\infty\},$$

$$D^+(x,f) = \{(y,g) : \pi(x_i,f_i;t_i) \to (y,g) \text{ for some sequences } (x_i,f_i) \to (x,f) \text{ and } t_i \geq 0\}, \text{ and}$$

$$J^+(x,f) = \{(y,g) : \pi(x_i,f_i;t_i) \to (y,g) \text{ for some sequences } (x_i,f_i) \to (x,f) \text{ and } t_i \to +\infty\}.$$

$\Omega(x,f)$ is called the positive limit set, $D^+(x,f)$ the positive prolongation, and $J^+(x,f)$ the positive prolongational limit set of (x,f). The negative limit set $A(x,f)$, the negative prolongation $D^-(x,f)$, and the negative prolongational limit set $J^-(x,f)$ are defined similarly.

Now consider the projection mapping

$$P : W \times F^*_{co} \to W.$$

Definition 5.2.3.

Define

$$\Omega_\phi(x,f) = P(\Omega(x,f)),\ D_\phi^+(x,f) = P(D^+(x,f)), \text{ and}$$

$$J_\phi^+(x,f) = P(J^+(x,f)).$$

$\Omega_\phi(x,f)$ is called the positive limit set, $D_\phi^+(x,f)$ the positive prolongation, and $J_\phi^+(x,f)$ the positive prolongational limit set of the solution $\phi(x,f,t)$.

Remark 5.2.1.

It is clear that for any $(x,f) \in W \times F_{co}^*$, $D_\phi^+(x,f) = \gamma_\phi^+(x,f) \cup J_\phi^+(x,f)$ and $\Omega_\phi(x,f) \subsetneq J_\phi^+(x,f)$, where $\gamma_\phi^+(x,f) = P(\gamma^+(x,f))$.

It is proved in [44] that if the motion f_t in F_{co}^* is positively compact, then $\Omega_\phi(x,f)$ can be characterized as follows.

Lemma 5.2.1 [44, Lemma 2].

Let $f \in C(W \times R, R^n)$ be a regular function and assume that the motion f_t is positively compact. Then a point $\hat{x}$ lies in the positive limit set $\Omega_\phi(x,f)$ if and only if there is a sequence $\{\tau_n\}$ in R with $\tau_n \to +\infty$ and $\phi(x,f,\tau_n) \to \hat{x}$.

Similar characterizations for $D_\phi^+(x,f)$ and $J_\phi^+(x,f)$ are given in the following two lemmas.

Lemma 5.2.2.

Let $f \in C(W \times R, R^n)$ be a regular function and assume that the motion f_t is positively compact. Then a point $\hat{x}$ lies in

$D^+_\phi(x,f)$ if and only if $\phi(x_n,f,\tau_n) \to \hat{x}$ for some sequences $x_n \to x$, $\tau_n \geq 0$.

Proof. Let $\hat{x} \in D^+_\phi(x,f)$. By the definition of $D^+_\phi(x,f)$, there is an $\hat{f}$ in F^*_{co} such that $(\hat{x},\hat{f}) \in D^+(x,f)$. Further, there are sequences $\{(y_n,g_n)\}$ and $\{T_n\}$ such that

$$(y_n,g_n) \to (x,f),\ T_n \geq 0,\ \text{and}\ \pi(y_n,g_n;T_n) \to (\hat{x},\hat{f}).$$

This implies that there are sequences $\{y_n\}$, $\{g_n\}$ and $\{T_n\}$ such that

$$y_n \to x,\ g_n \to f,\ T_n \geq 0\ \text{ and }\ \phi(y_n,g_n,T_n) \to \hat{x}\ .$$

Now we can assume that the sequence $\{g_n\}$ is of the form $\{f_{t_n}\}$, where $t_n \to 0$. Let $\phi(y_n,f_{t_n}, -t_n) = x_n$. As $y_n \to x$ and $t_n \to 0$, we have $x_n \to x$ and $\phi(x_n,f,T_n+t_n) \to \hat{x}$. Hence, if $\hat{x} \in D^+_\phi(x,f)$ then there are sequences $x_n \to x$ and $\tau_n = t_n + T_n \geq 0$ such that $\phi(x_n,f,\tau_n) \to \hat{x}$.

Conversely, suppose that $\phi(x_n,f,\tau_n) \to \hat{x}$ for some sequences $x_n \to x$ and $\tau_n \geq 0$. Consider the sequence $\{f_{\tau_n}\}$. Since f_t is positively compact, there exists a convergent subsequence $\{f_{\tau_{n_k}}\}$ of $\{f_{\tau_n}\}$. Let $f_{\tau_{n_k}} \to \hat{f} \in F^*_{co}$. Thus, we have

$$x_{n_k} \to x,\ \tau_{n_k} \geq 0,\ \phi(x_{n_k},f,\tau_{n_k}) \to \hat{x}\ \text{and}\ f_{\tau_{n_k}} \to \hat{f}.$$

This implies that $\pi(x_{n_k}, f\,;\,\tau_{n_k}) \to (\hat{x},\hat{f})$ for some sequences $x_{n_k} \to x$ and $\tau_{n_k} \geq 0$. Therefore $(\hat{x},\hat{f}) \in D^+(x,f)$, which implies that $\hat{x} \in D^+_\phi(x,f)$. This completes the proof.

Lemma 5.2.3.

Let $f \in C(W \times R, R^n)$ be a regular function and assume that the motion f_t is positively compact. Then a point $\hat{x}$ lies in $J_\phi^+(x,f)$ if and only if $\phi(x_n,f,\tau_n) \to \hat{x}$ for some sequences $x_n \to x$ and $\tau_n \to +\infty$.

The proof is similar to the proof of Lemma 5.2.2 and hence omitted.

Now we know that $\gamma_\phi^+(x,f) \subset D_\phi^+(x,f)$ and $\Omega_\phi(x,f) \subset J_\phi^+(x,f)$. These inclusions may be proper and can be shown by the following simple example.

Example 5.2.1.

Consider the differential system

$$\dot{x}_1 = \frac{-x_1}{(t+2)\log(t+2)}, \quad \dot{x}_2 = \frac{x_2}{(t+2)\log(t+2)}, \quad (t \geq 0),$$

in R^2. The general solution is

$$x(t) = (x_1(t), x_2(t)) = \left(\frac{A}{\log(t+2)}, B\log(t+2)\right),$$

where A and B are arbitrary constants. The trajectories are shown in Fig. 5.2.1. For any point $P = (x_1, 0)$, $D_\phi^+(P) = \gamma_\phi^+(P) \cup \{(x_1,x_2) : x_1 = 0\}$, $J_\phi^+(P) = \{(x_1,x_2) : x_1 = 0\}$, and $\Omega_\phi(P) = \{(0,0)\}$. Clearly, $D_\phi^+(P) \neq C\ell[\gamma_\phi^+(P)]$, and $J_\phi^+(P) \neq \Omega_\phi(P)$.

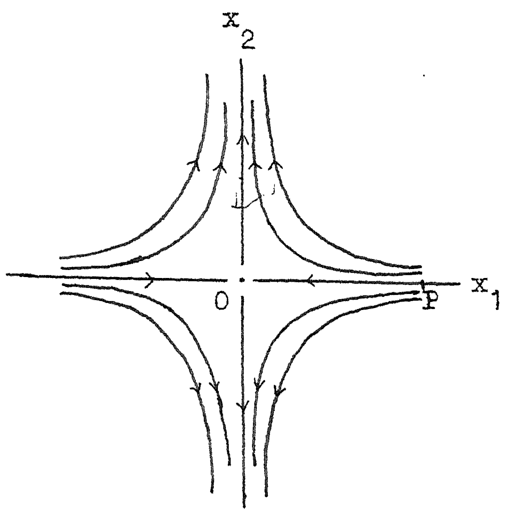

Fig. 5.2.1.

5.3 COMPACT SOLUTIONS

Now we shall obtain some theorems which relate the behavior of compactness property of solutions of (4.2.1) with the corresponding behavior of solutions of (4.2.2).

Theorem 5.3.1.

Let $f \in C(W \times R, R^n)$ be a regular function and assume that the motion f_t is positively compact. Let $x \in W$ and N be a neighborhood of x such that

$$C\ell\ [\bigcup_{\tau \in R} \{\phi(y,f_\tau,t) : y \in N,\ t \geq 0\}]$$

is a compact subset of W. Then the sets $D^+(x,f)$ and $J^+(x,f)$ are nonempty, compact and connected. If $(x^*,f^*) \in J^+(x,f)$, then the solution $\phi(x^*,f^*,t)$ of (4.2.2) is compact. Furthermore, $D^+_\phi(x,f)$ and $J^+_\phi(x,f)$ are nonempty, compact and connected.

Proof. Let N be a neighborhood of x such that

$$C\ell\ [\bigcup_{\tau \in R} \{\phi(y,f_\tau,t) : y \in N,\ t \geq 0\}]$$

be a compact subset of W. Consider the set $N \times F$, where $F = \{f_\tau : \tau \in R\}$. Obviously $N \times F$ is a neighborhood of (x,f). Further, we have

$$C\ell[\pi(N \times F;R^+)] \subset C\ell[\bigcup_{\tau \in R} \{\phi(y,f_\tau,t) : y \in N,\ t \geq 0\}] \times F^*_{co},$$

where $\pi(N \times F;\ R^+)$ denotes the set $\{\pi(x,f;t) : (x,f) \in N \times F,\ t \geq 0\}$. Since the motion f_t is positively compact, F^*_{co} is compact and hence $C\ell[\pi(N \times F;\ R^+)]$ is compact. Now the application of Theorem 2.4.1 yields that the sets $D^+(x,f)$ and $J^+(x,f)$ are nonempty, compact and connected. The rest of the proof follows from the fact that the projection mapping $P: W \times F^*_{co} \to W$ is continuous.

Remark 5.3.1.

G.R. Sell [44, Theorem 1] assumed that the solution $\phi(x,f,t)$ of (4.2.1) is positively compact to prove a similar result for ω-limit set $\Omega_\phi(x,f)$. Example 5.2.1 shows that the assumption of positive compactness of the solution $\phi(x,f,t)$ is not sufficient to conclude Theorem 5.3.1. For, it is obvious (Fig. 5.2.1) that for any $P = (x_1,0) \in R^2$, where $x_1 \neq 0$, $C\ell\ [\gamma^+_\phi(P)]$ is compact but $J^+_\phi(P) = \{(x_1,x_2): x_1 = 0,\ x_2 \in R\}$ is not compact.

Theorem 5.3.2.

Let $f \in C(W \times R,\ R^n)$ be a regular function and assume that the motion f_t is positively compact. Let $D^+(x,f)$ be a compact subset of $W \times F^*_{co}$. Then for every point $(x^*,f^*) \in J^+(x,f)$, the solution $\phi(x^*,f^*,t)$ of (4.2.2) is compact. Moreover, there exist sequences $\{x_n\}$ in W and $\{\tau_n\}$ in R with $x_n \to x$, $\tau_n \to +\infty$

such that $\phi(x_n, f, \tau_n + t)$ converges to $\phi(x^*, f^*, t)$ uniformly on compact sets in R.

Proof. The first part follows from Theorem 5.3.1. For the second part, since $\phi(x^*, f^*, t)$ is compact, it follows that $I_{(x^*, f^*)} = R$. Therefore, the solution $\phi(x^*, f^*, t)$ and the motion $\pi(x^*, f^*; t)$ are defined for all t in R. Now let $\{x_n\}$ and $\{\tau_n\}$ be sequences in W and R respectively, with $x_n \to x$, $\tau_n \to +\infty$ and $\pi(x_n, f; \tau_n) \to (x^*, f^*)$. Since π is continuous, $\pi(x_n, f; \tau_n + t)$ converges to $\pi(x^*, f^*; t)$ uniformly on compact sets in R. (see Lemma 2.3.5). Further, since the projection mapping P is continuous, the solutions

$$\phi(x_n, f, \tau_n + t) = P(\pi(x_n, f; \tau_n + t))$$

converge to $\phi(x^*, f^*, t)$ uniformly on compact sets in R. This completes the proof.

The conclusion of Theorem 5.3.2 can be formulated in terms of the positive prolongational limit set $J_\phi^+(x,f)$. The basic fact here is that if $x^* \varepsilon J_\phi^+(x,f)$, then there is an f^* in F_{co}^* such that $(x^*, f^*) \varepsilon J^+(x,f)$.

Corollary 5.3.1.

Let $f \varepsilon C(W \times R, R^n)$ be a regular function and assume that the motion f_t is positively compact. Let $D^+(x,f)$ be a compact subset of $W \times F_{co}^*$. Then for every point $x^* \varepsilon J_\phi^+(x,f)$, there is a function f^* in $\Omega^*(f)$ such that the solution $\phi(x^*, f^*, t)$ of (4.2.2) is compact. Moreover, there exist sequences $x_n \to x$,

$\tau_n \to +\infty$ such that

$$\phi(x_n, f, \tau_n + t) \to \phi(x^*, f^*, t)$$

uniformly on compact sets in R.

This corollary asserts that the positive prolongational limit set $J_\phi^+(x,f)$ is quasi-invariant in the sense defined by R.K. Miller [35] .

If f is asymptotically autonomous, that is, if $\Omega^*(f)$ is a singleton, then one can say more.

Corollary 5.3.2.

Let $f \in C(W \times R, R^n)$ be a regular function that is asymptotically autonomous. Let $\Omega^*(f) = \{f^*\}$. Then every positive prolongational limit set $J^+(x,f)$ can be expressed in the form

$$J^+(x,f) = J_\phi^+(x,f) \times \{f^*\} \quad .$$

Therefore, $J_\phi^+(x,f)$ is the union of solutions of

$$x' = f^*(x).$$

If $D^+(x,f)$ is compact, then for every x^* in $J_\phi^+(x,f)$ the solution $\phi(x^*,f^*,t)$ is compact, and there exist sequences $x_n \to x$, $\tau_n \to +\infty$ such that

$$\phi(x_n, f, \tau_n + t) \to \phi(x^*, f^*, t)$$

uniformly on compact sets in R.

Theorem 5.3.3.

Let $f \in C(W \times R, R^n)$ be a regular function and assume that the motion f_t is positively compact. Let $\Omega_\phi(x,f)$ be a nonempty

compact subset of W. Then the solution $\phi(x,f,t)$ of (4.2.1) is positively compact.

Proof. Consider the set $\Omega(x,f)$. Since $\Omega(x,f)$ is a closed subset of the compact set $\Omega_\phi(x,f) \times F^*_{co}$, it is compact. This implies that the closure of $\gamma^+(x,f)$ is compact (Lemma 2.4.1). Further, since the projection mapping P is continuous, it follows that the solution $\phi(x,f,t)$ of (4.2.1) is positively compact.

Theorem 5.3.4.

Let $f \in C(W \times R, R^n)$ be a regular function and assume that the motion f_t is positively compact. Then $\Omega_\phi(x,f)$ is nonempty and compact whenever $J^+_\phi(x,f)$ is nonempty and compact.

Proof. Let $J^+_\phi(x,f)$ be a nonempty and compact subset of W. Then the set $J^+(x,f)$, being a closed subset of the compact set $J^+_\phi(x,f) \times F^*_{co}$, is nonempty and compact. Now the application of Lemma 2.4.2 yields that $D^+(x,f)$ is compact. Since $C\ell[\gamma^+(x,f)] \subset D^+(x,f)$, $C\ell[\gamma^+(x,f)]$ is compact, and therefore by Lemma 2.4.1, $\Omega(x,f)$ is nonempty and compact. The rest of the proof follows from the fact that the projection mapping P is continuous.

5.4 STABILITY THEOREMS

In this section we shall give some results which characterize the stability properties of solutions of (4.2.1) in terms of its prolongation and prolongational limit sets.

for $n \geq N$. This shows that $\phi(x,f,\tau_n) \to y$. Therefore by Lemma 5.2.1, $y \in \Omega_\phi(x,f)$. Hence

$$D_\phi^+(x,f) = C\ell[\gamma_\phi^+(x,f)].$$

This completes the proof.

Theorem 5.4.1.

Let $f \in C(W \times R, R^n)$ be a regular function with $f(0,t) = 0$ for all $t \geq 0$ and assume that the motion f_t is positively compact. Then the null solution of (4.2.1) is stable if and only if

$$D_\phi^+(0,f) = \{0\} .$$

The necessity part of the proof follows from Lemma 5.4.1 and sufficiency part is direct and hence omitted.

Corollary 5.4.1.

Let $f \in C(W \times R, R^n)$ be a regular function with $f(0,t) = 0$ for all $t \geq 0$ and assume that the motion f_t is positively compact. Then the null solution of (4.2.1) is stable if the local dynamical system π on $W \times F_{co}^*$ is stable.

Proof. The stability of π implies that for each $(x,f) \in W \times F_{co}^*$,

$$D^+(x,f) = C\ell[\gamma^+(x,f)]$$

(see Theorem 2.4.2). Therefore $D^+(0,f) = C\ell[\gamma^+(0,f)]$ and by taking the projection P of both the sets on W, we have

$$D_\phi^+(0,f) = \{0\} .$$

Thus the application of Theorem 5.4.1 yields the desired result.

Theorem 5.4.2.

Let $f \in C(W \times R, R^n)$ be a regular function with $f(0,t) = 0$ for all $t \geq 0$ and assume that the motion f_t is positively compact. Then the null solution of (4.2.1) is asymptotically stable if and only if $\{x_o : J^+_\phi(x_o,f) = \{0\}\}$ is a neighborhood of the origin.

Proof. Let the null solution of (4.2.1) be asymptotically stable. Hence it is stable. Therefore by Theorem 5.4.1, we have

$$D^+_\phi(0,f) = \{0\}.$$

Further, we have for a given $\delta_o > 0$ and for each $\eta > 0$ there exists a $T = T(\eta) > 0$ such that whenever $|x_o| < \delta_o$,

$$|\phi(x_o,f,t)| < \eta$$

for all $t \geq T$. This implies that $J^+_\phi(x_o,f) = \{0\}$, because by taking sequences $x_n \to x_o$ and $\tau_n \to +\infty$, one can easily prove that $\phi(x_n,f,\tau_n) \to 0$. Hence

$$\{x_o : |x_o| < \delta_o\}$$

is the neighborhood of the origin such that $J^+_\phi(x_o,f) = \{0\}$.

Conversely, suppose that

$$\{x_o : J^+_\phi(x_o,f) = \{0\}\}$$

is a neighborhood of the origin. This shows that $J^+_\phi(0,f) = \{0\}$. Further, we have

$$D^+_\phi(0,f) = \gamma^+_\phi(0,f) \cup J^+_\phi(0,f) = \{0\}.$$

Hence, by Theorem 5.4.1, it follows that the null solution of (4.2.1)

is stable. For the rest of the proof, let δ_0 be a positive real number such that

$$\{x_0 : |x_0| < \delta_0\} \subset \{x_0 : J_\phi^+(x_0,f) = \{0\}\}.$$

Then since for each x_0 satisfying $|x_0| < \delta_0$, $J_\phi^+(x_0,f)$ is a nonempty compact set, we have by Theorem 5.3.4 that the set $\Omega_\phi(x_0,f)$ is nonempty. Therefore $\Omega_\phi(x_0,f) = \{0\}$ whenever $|x_0| < \delta_0$. This implies that for a given $\eta > 0$ there exists a $T = T(\eta) > 0$ such that whenever $|x_0| < \delta_0$,

$$|\phi(x_0,f,t)| < \eta$$

for all $t \geq T$. This completes the proof.

CHAPTER 6

LOCAL DYNAMICAL SYSTEMS AND EXTENSION OF LYAPUNOV'S METHOD

6.1 INTRODUCTION

It has long been realized that the theory of topological dynamics has direct and important applications to autonomous differential equations, but this theory is not well developed as a powerful technique in applications to nonautonomous differential equations. However, quite recently G.R. Sell [43] has shown that there is a way of viewing the solutions of nonautonomous differential equations as a dynamical system. This view point is very general and includes all differential equations satisfying only the weakest hypotheses. In particular, he has shown that the solutions of every admissible differential equation $x' = f(x,t)$, defined on $W \times R$, can be viewed as a local dynamical system π defined on the phase space $W \times F^*_{co}$, where W is an open set in R^n and F^*_{co} is the 'hull' of f.

Our aim in this chapter is to extend the phase space of the local dynamical system π by defining the prolongational set D^*_f, and investigate sufficient conditions for stability of solutions of the given differential equation and the corresponding set of limiting equations. This would generalize most of the results of G.R. Sell in [43] and [44], as the phase space in this case contains $W \times F^*_{co}$.

Section 6.2 deals with preliminaries and basic lemmas. Section 6.3 is devoted for the extension of local dynamical system π on $W \times F^*_{co}$ to one on $W \times D^*_f$. In Section 6.4, we obtain sufficient conditions to establish a number of results on stability of solutions of the given differential equation and the corresponding set of limiting equations in terms of Lyapunov functions. A simple example is constructed to illustrate the results.

6.2 DEFINITIONS AND BASIC LEMMAS

Throughout this chapter we follow the same notations as in Chapter 4.

Let $C^*(W \times R, R^n)$ be the class of all admissible functions f defined on $W \times R$ with values in R^n. It is known [43, Theorem 1] that the mapping

$$\pi^* : C \times R \to C,$$

defined by $\pi^*(f,t) = f_t$, is a dynamical system on C, when C has the compact open topology. Therefore, it is evident that

$$\pi^* : C^* \times R \to C^*,$$

the restriction of π^* on $C^* \subset C$ is also a dynamical system on C^*, when C^* has the subspace topology. Let ρ be a metric which generates the compact open topology on C.

Definition 6.2.1.

Define positive prolongation $D^+(f)$ and positive prolongational limit set $J^+(f)$ of $f \in C^*$ as follows :

$D^+(f) = \{g \in C:$ there is a sequence $\{f_n\}$ in C^* and a sequence $\{t_n\}$ in R^+ such that $f_n \to f$ and $\pi^*(f_n,t_n) \to g\}$,

$J^+(f) = \{g \in C:$ there is a sequence $\{f_n\}$ in C^* and a sequence $\{t_n\}$ in R^+ such that $f_n \to f$, $t_n \to +\infty$, and $\pi^*(f_n,t_n) \to g\}$.

Similarly, we define negative prolongation $D^-(f)$ and negative prolongational limit set $J^-(f)$ of f by considering the sequence $\{t_n\}$ in R^-. Let $D_f^* = D^+(f) \cup D^-(f)$.

Definition 6.2.2.

We say that the function $f \in C^*$ is D_f^*-regular if and only if every $g \in D_f^*$ is admissible.

Remark 6.2.1.

It is clear that, for any $f \in C^*$, the ω-limit set $\Omega^*(f) \subset J^+(f)$ and $D^+(f) = F \cup J^+(f)$, where $F = \{f_\tau : \tau \in R^+\}$. Further, if f is D_f^*-regular, then f is regular.

We need the following lemmas in our subsequent discussion.

Lemma 6.2.1.

Given $h \in J^+(f)$ and $\varepsilon > 0$, there exist a function $g \in C^*$ and a real number $\tau = \tau(\varepsilon) > 0$ such that

$$\rho(f,g) < \varepsilon \text{ and } \rho(h,g_\tau) < \varepsilon.$$

Proof. Let $h \in J^+(f)$ and $\varepsilon > 0$ be given. From the definition of $J^+(f)$, there exist sequences $\{f_n\} \in C^*$, $\{t_n\} \in R^+$, such that

$$f_n \to f,\ t_n \to +\infty, \text{ and } \pi^*(f_n, t_n) \to h.$$

Therefore, it follows that $\rho(f, f_n) < \varepsilon$ for all $n \geq N_1(\varepsilon)$, and $\rho(h, \pi^*(f_n, t_n)) < \varepsilon$ for all $n \geq N_2(\varepsilon)$. Let $N = \max(N_1, N_2)$. Choose $f_N = g$ and $t_N = \tau$. Then obviously $g \varepsilon C^*$, $\rho(f,g) < \varepsilon$, and $\rho(h, g_\tau) < \varepsilon$. This completes the proof.

Lemma 6.2.2.

If $f \varepsilon C^*$, then

(i) $J^+(f)$ is a closed and invariant subset of C, and

(ii) $D^+(f)$ is a closed and positively invariant subset of C.

Proof. (i) Let $\{g_n\}$ be a sequence in $J^+(f)$ with $g_n \to g$. We prove that $g \varepsilon J^+(f)$. By definition, for each integer k, there are sequences $\{f_n^k\} \varepsilon C^*$ and $\{t_n^k\} \varepsilon R^+$ such that

$$f_n^k \to f,\ t_n^k \to +\infty, \text{ and } \pi^*(f_n^k, t_n^k) \to g_k.$$

We may assume by taking subsequences if necessary that

$$t_n^k > k,\ \rho(f_n^k, f) \leq \frac{1}{k}, \text{ and } \rho(\pi^*(f_n^k, t_n^k), g_k) \leq \frac{1}{k}$$

for $n \geq k$. Now consider the sequences $\{f_n^n\}$ and $\{t_n^n\}$. Obviously $\{f_n^n\} \varepsilon C^*$ and $f_n^n \to f$, $t_n^n \to +\infty$. Hence

$$\rho(g, \pi^*(f_n^n, t_n^n)) \leq \rho(g, g_n) + \rho(g_n, \pi^*(f_n^n, t_n^n)) \leq \rho(g, g_n) + \frac{1}{n}.$$

Since $g_n \to g$, we have $\pi^*(f_n^n, t_n^n) \to g$. Thus $g \varepsilon J^+(f)$. This shows that $J^+(f)$ is closed in C.

To prove that $J^+(f)$ is invariant, let $g \varepsilon J^+(f)$ and $\tau \varepsilon R$ be given. Then, there exist sequences $\{f_n\} \varepsilon C^*$ and $\{t_n\} \varepsilon R^+$ such that

$$f_n \to f, \; t_n \to +\infty, \text{ and } \pi^*(f_n,t_n) \to g \; .$$

Now consider the sequence $\{t_n + \tau\}$. Clearly $t_n + \tau \to +\infty$, and

$$\pi^*(f_n, t_n+\tau) = \pi^*(\pi^*(f_n, t_n), \tau) = \pi^*(g, \tau).$$

Since $f_n \to f$, we have $\pi^*(g,\tau) \in J^+(f)$. As $\tau \in R$ is arbitrary, it is clear that $J^+(f)$ is invariant.

The proof of (ii) is similar and hence omitted.

Remark 6.2.2.

Analogous results hold for $D^-(f)$ and $J^-(f)$.

6.3 EXTENSION OF LOCAL DYNAMICAL SYSTEM

Let $f \in C^*$ and $F = \{f_\tau : \tau \in R\}$. For each point $p = (x,g)$ in $X = W \times F$, let $I_p = I_{(x,g)}$ be the maximal interval of definition of the solution $\phi(x,g,t)$ of $x' = g(x,t)$ that satisfies $\phi(x,g,0) = x$. Let

$$S = \{(x,g;t) = (p;t) \in X \times R : t \in I_p\} \quad ,$$

and define $\pi : S \to X$ by

$$\pi(x,g;t) = (\phi(x,g,t), g_t). \tag{6.3.1}$$

It has been proved by G.R. Sell in [43] that π defined by (6.3.1) is a local dynamical system on $X = W \times F$.

Theorem 6.3.1.

Let $f \in C^*(W \times R, R^n)$ be a function that satisfies a local Lipschitz condition in x, where the Lipschitz constant is independent of t. Then f is D_f^*-regular.

The Lipschitz condition stated above means that for every compact subset K of W, there is a positive constant c such that

$$|f(x,t) - f(y,t)| \le c\,|x-y| \quad (x \in K, y \in K, t \in R).$$

Proof. Let K be a compact set in W and let f_τ be any translate of f. Then there is a positive constant c such that

$$\begin{aligned} |f_\tau(x,t) - f_\tau(y,t)| &= |f(x,t+\tau) - f(y,t+\tau)| \\ &\le c\,|x-y|, \end{aligned}$$

for all x and y in K and all t in R. In other words, every translate of f satisfies the same Lipschitz condition. Now let $h \in J^+(f)$. Then by Lemma 6.2.1, given $\varepsilon > 0$ there exist a function $g \in C^*$ and a real number $\tau = \tau(\varepsilon) > 0$ such that

$$\rho(f,g) < \varepsilon \text{ and } \rho(h,g_\tau) < \varepsilon .$$

Let I be any compact set in R and let $I' = \{t+\tau : t \in I\}$. Then $M = K \times I$ and $M' = K \times I'$ are compact sets in $W \times R$. Since the metric ρ generates the topology of uniform convergence on compact sets (this topology is the same as the compact open topology on $C(W \times R, R^n)$ [28, pp. 186 and 230]), there is a nonnegative function $\ell(M'; \varepsilon)$ such that $\ell(M'; \varepsilon) \to 0$ as $\varepsilon \to 0$, and

$$\sup\{|f(x,t)-g(x,t)| : (x,t) \in M'\} \le \ell(M'; \varepsilon),$$

whenever $f \in C(W \times R, R^n)$ and $\rho(f,g) < \varepsilon$. (The function ℓ depends on f, which is fixed for our argument). Thus if $\rho(f,g) < \varepsilon$, then

$$\begin{aligned} |g(x,t)-g(y,t)| &\le |g(x,t)-f(x,t)| + |f(x,t) - f(y,t)| + |f(y,t) - g(y,t)| \\ &\le 2\ell(M';\varepsilon) + c\,|x-y|, \end{aligned}$$

whenever (x,t) and (y,t) are in M'. This implies that

$$|g_\tau(x,t) - g_\tau(y,t)| \leq 2\ell(M';\varepsilon) + c\,|x-y|,$$

whenever (x,t) and (y,t) are in M. Now since $\rho(h,g_\tau) < \varepsilon$, we can again construct a nonnegative function $\ell'(M;\varepsilon)$, such that $\ell'(M;\varepsilon) \to 0$ as $\varepsilon \to 0$, and

$$\sup\{|h(x,t) - g_\tau(x,t)| : (x,t) \in M\} \leq \ell'(M;\varepsilon).$$

Therefore, we have

$$|h(x,t)-h(y,t)| \leq |h(x,t)-g_\tau(x,t)|+|g_\tau(x,t)-g_\tau(y,t)|+|g_\tau(y,t)-h(y,t)|$$

$$\leq 2\ell'(M;\varepsilon)+2\ell(M';\varepsilon) + c\,|x-y|,$$

whenever (x,t) and (y,t) are in M. If we let $\varepsilon \to 0$ we get

$$|h(x,t) - h(y,t)| \leq c|x-y| \quad (x \in K,\ y \in K,\ t \in I).$$

However, since I is arbitrary we get

$$|h(x,t) - h(y,t)| \leq c\,|x-y| \quad (x \in K,\ y \in K,\ t \in R).$$

Similarly, we can prove that every function h in $J^-(f)$ satisfies the same Lipschitz condition as f. This completes the proof.

Theorem 6.3.2.

Let $f \in C^*$ be a D_f^*-regular function. Then the local dynamical system π on $W \times F$ defined by (6.3.1), can be extended to a local dynamical system on $W \times D_f^*$. The extension is given by (6.3.1).

Proof. . Since every $h \in D_f^*$ is admissible, we can define the extension π by (6.3.1). It then follows that π satisfies all the properties for a local dynamical system (cf. Definition 2.3.6).

6.4 LYAPUNOV FUNCTIONS AND STABILITY

Let $f \in C^*$. If the positive prolongational limit set $J^+(f)$ is non-empty, then we say that the set of limiting equations for (4.2.1) is the set of all differential equations of the form

$$x' = f^*(x,t), \tag{6.4.1}$$

where $f^* \in J^+(f)$.

In our subsequent discussion, we suppose that $x(t) = x(x_o,f,t)$ and $y(t) = y(y_o,f^*,t)$ are any two solutions of (4.2.1) and (6.4.1) with $x(t_o) = x_o$ and $y(t_o)=y_o$, existing for all $t \in R^+$. We shall write $d(x,y)$ for $|x-y|$, Let a function $v(t,x,y) \geq 0$ be defined and continuous on the product space $R^+ \times W \times W$, and suppose that it satisfies local Lipschitz condition in x and y. Following Yoshizawa [50], we define the function

$$D^+v(t,x,y) = \overline{\lim_{h\to 0^+}} \frac{1}{h} [v(t+h,x+hf(x,t),y+hf^*(y,t))-v(t,x,y)] .$$

With respect to these functions we state the following lemma whose proof can be found in [29] .

Lemma 6.4.1.

Let the function $g(t,r)$ be defined and continuous on $R^+ \times R^+$. Suppose further that

$$D^+v(t,x,y) \leq g(t,v(t,x,y)).$$

Let $r(t)$ be the maximal solution of the differential equation

$$r' = g(t,r), \; r(t_o) = r_o \geq 0. \tag{6.4.2}$$

If $x(t)$ and $y(t)$ are any two solutions of (4.2.1) and (6.4.1) such that $v(t_0, x_0, y_0) \leq r_0$, then

$$v(t, x(t), y(t)) \leq r(t) \ , \ t \geq t_0 \ .$$

In order to unify our results on stability, we list the following conditions:

(c_1) For each $\varepsilon > 0$ and $t_0 \geq 0$, there exists a positive function $\delta = \delta(\varepsilon)$ such that if $d(x_0, y_0) \leq \delta$, then $d(x(t), y(t)) < \varepsilon$ for all $t \geq t_0$.

(c_2) For each $\eta > 0$ and $t_0 \geq 0$, there exist positive numbers δ_0 and $T = T(\eta)$ such that $d(x(t), y(t)) < \eta$ for $t \geq t_0 + T$, whenever $d(x_0, y_0) \leq \delta_0$.

(c_3) The conditions (c_1) and (c_2) hold simultaneously.

Remark 6.4.1.

Corresponding to the definitions above, if we say that the scalar differential equation (6.4.2) has the property ($c_1 s$), we mean that the following condition is satisfied:

($c_1 s$) Given $\varepsilon > 0$ and $t_0 \geq 0$, there exists a positive function $\delta = \delta(\varepsilon)$ such that the inequality $r_0 \leq \delta$ implies $r(t) < \varepsilon$ for all $t \geq t_0$.

Conditions (c_2) and (c_3) may be reformulated similarly.

Remark 6.4.2.

We assume hereafter that the solutions $r(t)$ of (6.4.2) are non-negative for $t \geq t_0$ so as to ensure that $g(t, r(t))$ is defined.

Such a requirement is clearly satisfied if we assume that $g(t,0) = 0$ for all t.

Further, we assume that

(A) the function $b(r)$ is continuous and nonincreasing in r, $b(r) > 0$ for $r > 0$, and $b(d(x,y)) \leq v(t,x,y)$.

Now we have the following main result on stability of solutions of (4.2.1) and (6.4.1).

Theorem 6.4.1.

Let the assumptions of Lemma 6.4.1 hold, together with (A). Suppose further that the scalar differential equation (6.4.2) satisfies one of the conditions $(c_1 s)$, $(c_2 s)$ and $(c_3 s)$, then the differential systems (4.2.1) and (6.4.1) satisfy the corresponding one of the conditions (c_1), (c_2) and (c_3).

The proof is similar to the proof of Theorem 4 in [29] and hence omitted.

Remark 6.4.3.

Similar to Theorem 6.4.1, some theorems are given in [29], [50] and [51] for two entirely different differential systems.

Example 6.4.1. (n = 1).

Consider the differential equation

$$x' = f(x,t), \tag{6.4.3}$$

where $f(x,t) = -x - x^5 e^{-2t}$.

Choose $f_n(x,t) = -x-x^5e^{-2t} - \frac{x^3t}{n}$, and $\{t_n\} = \{n\}$.

Then, we have

$$\pi^*(f_n, t_n) = -x-x^5e^{-2(t+n)} - \frac{x^3}{n}(t+n).$$

Therefore, $f^*(x,t) = -x-x^3$, where $f^* \varepsilon J^+(f)$. Let $v(t,x,y) = x^2+y^2$. Then the condition (A) is clearly satisfied. After a little computation, we obtain

$$D^+v(t,x,y) \leq -2\, v(t,x,y).$$

If we set $g(t,r) = -2r$, we see that the scalar differential equation (6.4.2) satisfies conditions (c_1s) and (c_2s). From Theorem 6.4.1, we conclude therefore that the solutions of the differential equation (6.4.3) and the corresponding limiting equation satisfy the conditions (c_1) and (c_2) and hence the condition (c_3).

BIBLIOGRAPHY

[1] AHMAD, S. : 'Strong attraction and classification of certain continuous flows', Math. Systems Theory, 5(1971), 157-163.

[2] AUSLANDER, J.; BHATIA, N.P.; SEIBERT, P.: 'Attractors in dynamical systems', Bol. Soc. Mat. Mexicana, 9(1964), 55-56.

[3] AUSLANDER, J.; SEIBERT, P. : 'Prolongations and stability in dynamical systems', Ann. Inst. Fourier (Grenoble), 14(1964), 237-267.

[4] BHATIA, N.P. : 'Stability and Lyapunov functions in dynamical systems', Contributions to Differential Equations, Vol. 3, New York, Wiley 1964, 175-188.

[5] BHATIA, N.P. : 'Weak attractors in dynamical systems', Bol. Soc. Mat. Mexicana, 11(1966), 56-64.

[6] BHATIA, N.P. : 'On asymptotic stability in dynamical systems', Math. Systems Theory, 1(1967), 113-128.

[7] BHATIA, N.P. : 'Attraction and non-saddle sets in dynamical systems', J. Diff. Eqs., 8(1970), 229-249.

[8] BHATIA, N.P.; HAJEK, O.: 'Local semi-dynamical systems', Lecture Notes in Mathematics, Vol. 90, Berlin-Heidelberg-New York, Springer 1969.

[9] BHATIA, N.P.; HAJEK, O. : 'Theory of dynamical systems', Part IV, Technical Note BN-610, June 1969, The Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, Maryland, U.S.A.

[10] BHATIA, N.P.; LAZER, A.C.; SZEGÖ, G.P.: 'On global weak attractors in dynamical systems', J. Math. Anal. Appl., 16(1966), 544-552.

[11] BHATIA, N.P.; SZEGÖ, G.P.: 'Dynamical Systems, Stability Theory and Applications', Lecture Notes in Mathematics, Vol. 35, Berlin-Heidelberg-New York, Springer 1967.

[12] BHATIA, N.P.; SZEGÖ, G.P.: 'Stability Theory of Dynamical Systems', Berlin-Heidelberg-New York, Springer 1970.

[13] BIRKHOFF, G.D. : 'Dynamical Systems', Amer. Math. Soc. Colloquium Publications, Vol. 9, New York 1927.

[14] BUSHAW, D.: 'A stability criterion for general systems', Math. Systems Theory, 1(1967), 79-88.

[15] CODDINGTON, E.A.; LEVINSON, N.: 'Theory of Ordinary Differential Equations', McGraw-Hill, New York, 1955.

[16] GOTTSCHALK, W.H.; Hedlund, G.A.: 'Topological Dynamics', Amer. Math. Soc. Colloquium Publications, Vol. 36, Providence 1955.

[17] HAHN, W.: 'Stability of Motion', Berlin-Heidelberg-New York, Springer 1967.

[18] HAJEK, O.: 'Dynamical Systems in the Plane', New York, Academic Press 1968.

[19] HAJEK, O.: 'Local characterization of local semi-dynamical systems', Math. Systems Theory, 2(1968), 17-26.

[20] HAJEK, O.: 'Compactness and asymptotic stability', Math. Systems Theory, 4(1970), 154-156.

[21] HAJEK, O.: 'Absolute stability of noncompact sets', J. Diff. Eqs., 9(1971), 496-508.

[22] HALE, J.K.: 'Dynamical systems and stability', J. Math. Anal. Appl., 26(1969), 39-59.

[23] IMDADI, S.M.S.; RAMA MOHANA RAO, M. : ' Dynamical systems in uniform spaces and Quasi-Lyapunov functions', To appear in Math. Systems Theory.

[24] IMDADI, S.M.S.; RAMA MOHANA RAO, M.: 'A note on Topological dynamics and Limiting equations', To appear in Proc. Amer. Math. Soc.

[25] IMDADI, S.M.S.; RAMA MOHANA RAO, M.: 'Topological dynamics, Prolongation and Prolongational limit sets', Communicated to J. Diff. Eqs.

[26] IMDADI, S.M.S.; RAMA MOHANA RAO, M.: 'Topological dynamics and extension of Lyapunov's second method', Communicated to J. Lond. Math. Soc.

[27] KAMKE, E.: 'Zur Theorie der Systeme gewöhnlicher Differentialgleichungen', II, Acta Math., 58(1932), 57-85.

[28] KELLEY, J.L.: 'General Topology', New York, Van Nostrand 1955.

[29] LAKSHMIKANTHAM, V.: 'Differential systems and extension of Lyapunov's method', Michigan Math. J., 9(1962), 311-320.

[30] LAKSHMIKANTHAN, V.; LEELA, S.: 'Differential and Integral Inequalities', I, New York-London, Academic Press 1969.

[31] LASALLE, J.P.; Lefschetz, S.: 'Stability by Liapunov's Direct Method with Applications', New York, Academic Press 1961.

[32] LEFSCHETZ, S.: 'Liapunov and stability in dynamical systems', Bol. Soc. Mat. Mexicana, 3(1958), 25-39.

[33] MARKOV, A.A.: 'Sur une propriété générale des ensembles minimaux de M. Birkhoff', C.R. Acad. Sci. Paris, 193(1931), 823-825.

[34] MARKUS, L.: 'Asymptotically autonomous differential systems', Contributions to the Theory of Nonlinear Oscillations, Vol. 3, Princeton, Princeton University Press 1956, 17-30.

[35] MILLER, R.K.: 'Asymptotic behavior of solutions of nonlinear differential equations', Trans. Amer. Math. Soc., 115(1965), 400-416.

[36] MILLER, R.K.; SELL, G.R.: 'Volterra integral equations and topological dynamics', Memoirs Amer. Math. Soc., 102(1970).

[37] NEMYTSKII, V.V.: 'Topological problems of the theory of dynamical systems', Amer. Math. Soc. Transl. no. 103, p. 85 (1954).

[38] NEMYTSKII, V.V.; STEPANOV, V.V.: 'Qualitative Theory of Differential Equations', Princeton, Princeton University Press 1960.

[39] ONUCHIC, N.: 'Relationships among the solutions of two systems of ordinary differential equations', Michigan J. Math., 10(1963), 129-139.

[40] ONUCHIC, N.: 'Applications of the topological method of Ważewski to certain problems of asymptotic behavior in ordinary differential equations', Pacific J. Math., 11(1961), 1511-1527.

[41] ROXIN, E.: 'Reachable zones in autonomous differential systems', Bol. Soc. Mat. Mexicana, 5(1960), 125-135.

[42] ROXIN, E.: 'On generalized dynamical systems defined by contingent equations', J. Diff. Eqs., 1(1965), 188-205.

[43] SELL, G.R.: 'Nonautonomous differential equations and topological dyanmics.I. The basic theory', Trans. Amer. Math. Soc., 127(1967), 241-262.

[44] SELL, G.R.: 'Nonautonomous differential equations and topological dynamics. II. Limiting equations', Trans. Amer. Math. Soc., 127(1967), 263-283.

[45] SELL, G.R.: 'Nonautonomous differential equations as dynamical systems', Proc. Intern. Symp. Differential Eqs. and Dynamical Systems, Puerto Rico, 1965, Academic Press, New York 1967, 531-536.

[46] URA, T.: 'Sur les courbes définies par les équations différentielles dans l'espace à m dimensions', Ann. Sic. École Norm. Sup., 70(1953), 287-360.

[47] URA, T.: 'On the flow outside a closed invariant set, stability, relative stability and saddle sets', Contributions to Differential Equations, Vol. 3, New York, Wiley 1964, 249-294.

[48] URA, T.; KIMURA, I.: 'Stability in topological dynamics', Proc. Japan Acad., 40(1964), 703-706.

[49] WHITNEY, H.: 'Regular families of curves', II, Proc. Nat. Acad. Sci. USA, 18(1932), 340-342.

[50] YOSHIZAWA, T.: 'Lyapunov's function and boundedness of solutions', Funkcial. Ekvac., 2(1959), 95-142.

[51] YOSHIZAWA, T.: 'Stability and boundedness of systems', Arch. Rational Mech. Anal., 6(1960), 409-421.

[52] YOSHIZAWA, T.: 'Asymptotic behavior of solutions of nonautonomous systems near sets', J. Math. Kyoto University, 1(1962), 303-323.

[53] YOSHIZAWA, T.: 'The stability theory by Liapunov's method', Math. Soc. Japan, Tokyo (1966).

[54] ZUBOV, V.I.: 'The Methods of Liapunov and their Applications', Leningrad 1964.